The Jones polynomial of almost-alternating and Turaev genus one links

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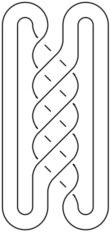
Construction of the Turaev surface $F(D)$

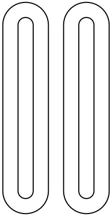
1. Replace arcs of D not near crossings with bands transverse to the projection plane.
2. Replace crossings of D with saddles interpolating between the all-B and all-A states of D
3. Cap off the boundary components with disks to obtain $F(D)$.

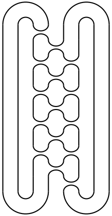
The Turaev surface in pictures



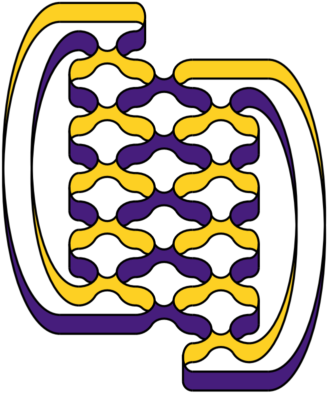
D

all-A state





all-B state



$F(D)$

Turaev genus

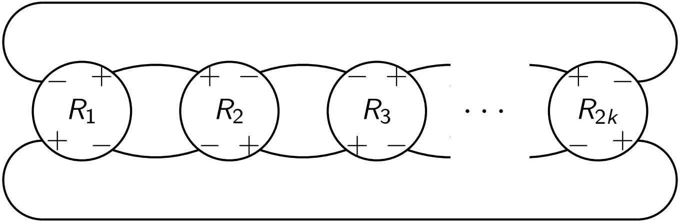
- For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface $F(D)$.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Some history of the Turaev surface

- Turaev (1987) related $F(D)$ to the difference between the span of the Jones polynomial and the crossing number.
- DFKLS (2006) - The Jones polynomial is an evaluation of the Bollobás-Riordan-Tutte polynomial of a graph embedded on the Turaev surface.
- CKS (2007), DL (2007, 2009, 2015) - Connections with Khovanov homology and knot Floer homology.
- Kalfagianni (2016) - Characterization of adequate knots using the colored Jones polynomial and Turaev genus.

Classification of links of Turaev genus one



Theorem (Armond, L.; Kim)

1. *Let $R_1, \ldots, R_{2k}$ be alternating two-tangles, and let D be a link diagram connecting $R_1, \ldots, R_{2k}$ as depicted above. Then $g_T(D) = 1$.*
2. *Moreover, if L is a non-split link with $g_T(L) = 1$, then L has a diagram as above.*

Examples of Turaev genus one links

Some examples of Turaev genus one links are:

- non-alternating pretzel links,
- non-alternating Montesinos links,
- almost-alternating links.

Almost-alternating links

- A link diagram D is almost-alternating if one crossing change transforms D into an alternating diagram (Adams et al. - 1992).
- A link L is almost-alternating if it is non-alternating and has an almost-alternating diagram.
- If L is almost-alternating, then $g_T(L) = 1$. The converse is open.

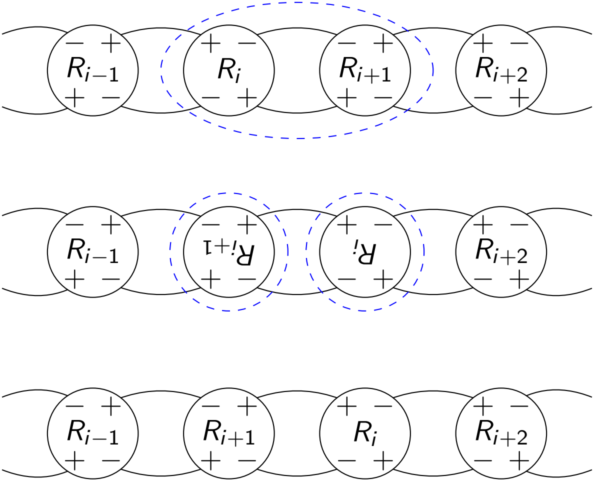
Mutation

- Let B be a 3-ball whose boundary intersects the link L in exactly 4-points. A mutation of L is a link obtained by removing B from S^3, rotating it 180° about a principle axis, and then gluing B back into S^3.
- Any two links related to one another via a sequence of mutations are said to be mutant to one another.

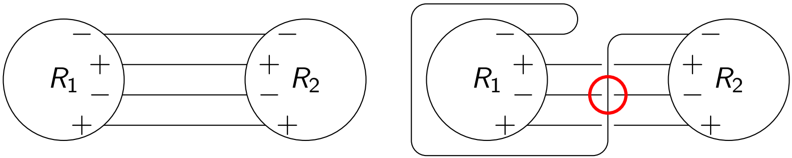
Theorem (Armond, L.)

If $g_T(L) = 1$, then L is mutant to an almost-alternating link.

Mutation proof



Mutation proof continued



Jones polynomial results

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

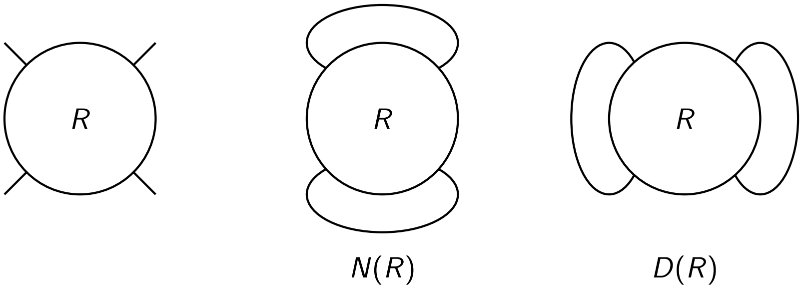
Theorem (Dasbach, L.)

If L is almost-alternating, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Corollary

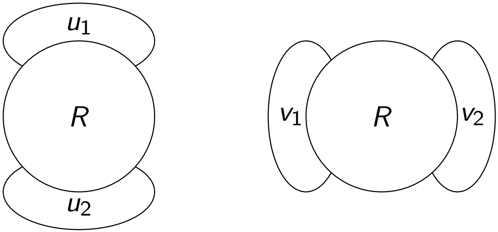
If $g_T(L) = 1$, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Numerator and denominator



A tangle R, its numerator closure $N(R)$ and its denominator closure $D(R)$.

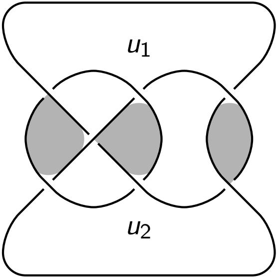
Adjacent faces



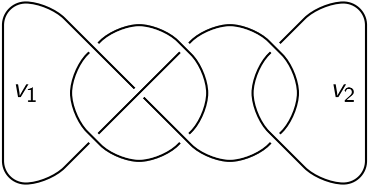
$N(R)$ $D(R)$

- Let $\mathrm{adj}(u_1, u_2)$ be the number of faces of $N(R)$ adjacent to both u_1 and u_2.
- Let $\mathrm{adj}(v_1, v_2)$ be the number of faces of $D(R)$ adjacent to both v_1 and v_2.

Example



$\mathrm{adj}(u_1, u_2) = 3$



$\mathrm{adj}(v_1, v_2) = 0$

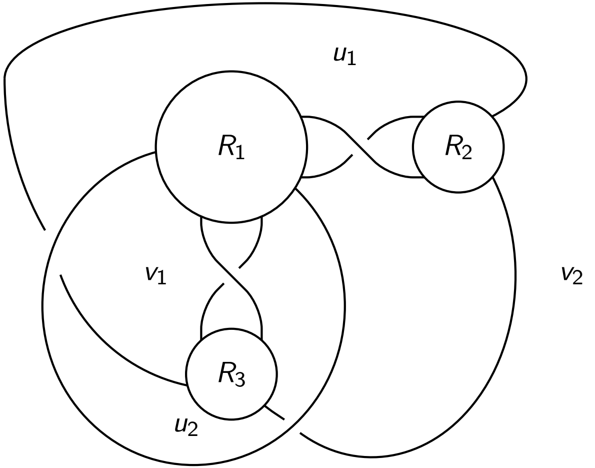
Non-minimal almost-alternating diagrams

Lemma

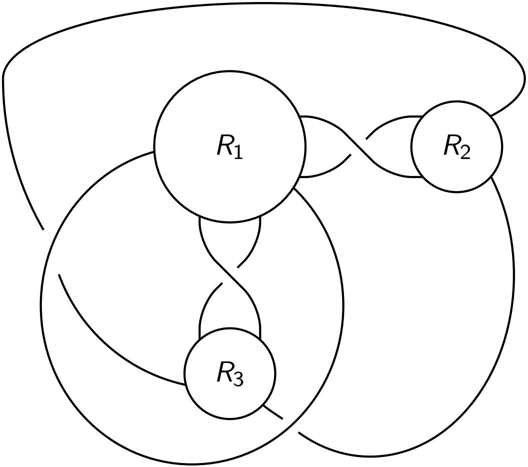
Let D be an almost-alternating diagram of a non-alternating link L. If $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$, then L has an almost-alternating diagram with fewer crossings than D.

Proof of Lemma

If $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$, then D has diagram as below.



Proof of Lemma



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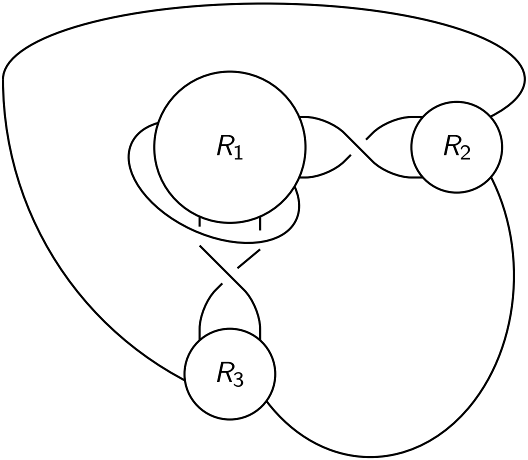
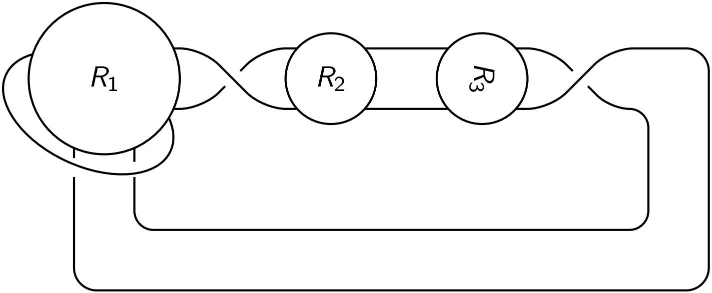


Proof of Lemma



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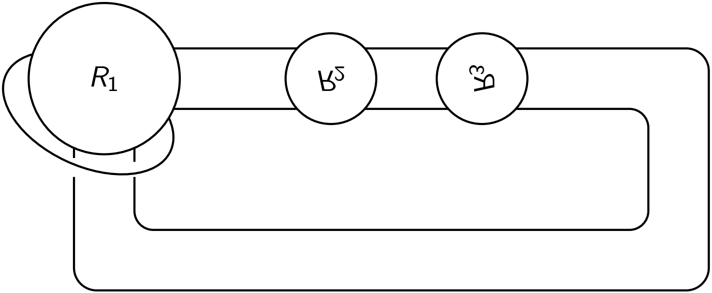

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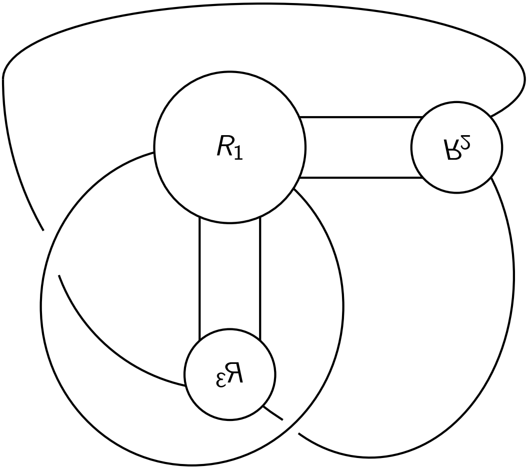
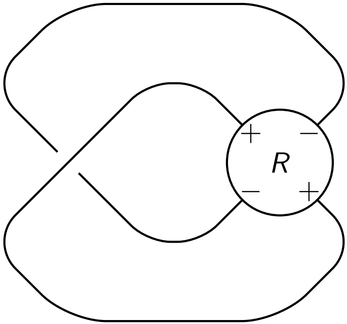


Resolutions of an almost-alternating diagram

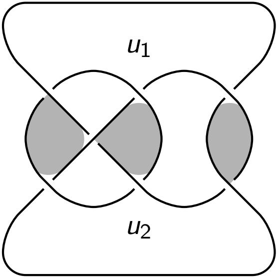
If D is almost-alternating, then the resolutions of the almost-alternating crossing $N(R)$ and $D(R)$ are alternating.



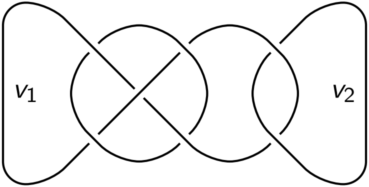
Jones polynomial proof

- $\langle D \rangle = A\langle D(R)\rangle + A^{-1}\langle N(R)\rangle$.
- Both $D(R)$ and $N(R)$ are alternating diagrams.
- Dasbach and Lin (2006) describe the first few coefficients of the Kauffman bracket of an alternating link in terms of the checkerboard graph.
- The (potential) leading and trailing coefficients of $\langle D \rangle$ are $|\operatorname{adj}(u_1, u_2) - 1|$ and $|\operatorname{adj}(v_1, v_2) - 1|$.

The example ... again



$\mathrm{adj}(u_1, u_2) = 3$



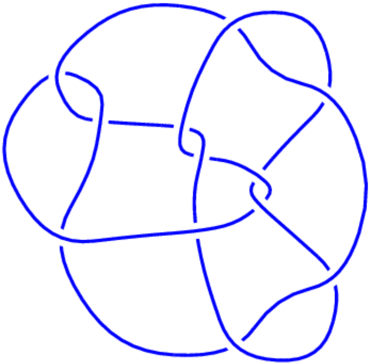
$\mathrm{adj}(v_1, v_2) = 0$

If $\mathrm{adj}(u_1, u_2) \geq 3$, then $\mathrm{adj}(v_1, v_2) = 0$, and if $\mathrm{adj}(v_1, v_2) \geq 3$, then $\mathrm{adj}(u_1, u_2) = 0$.

The leading and trailing coefficients

- We want either $|\operatorname{adj}(u_1, u_2) - 1| = 1$ or $|\operatorname{adj}(v_1, v_2) - 1| = 1$.
- Let D be an almost-alternating diagram of L with the fewest number of crossings among all almost-alternating diagrams of L.
- Lemma says at least one of $\operatorname{adj}(u_1, u_2)$ or $\operatorname{adj}(v_1, v_2)$ is not one. Say $\operatorname{adj}(u_1, u_2) \neq 1$.
- If $\operatorname{adj}(u_1, u_2) = 0$ or 2, and we are done. If $\operatorname{adj}(u_1, u_2) \geq 3$, then $\operatorname{adj}(v_1, v_2) = 0$, and we are done.

An example: $12n375$



$$V_{12n375}(t) = 2t^2 - 4t^3 + 8t^4 - 9t^5 + 10t^6 - 10t^7 + 7t^8 - 5t^9 + 2t^{10}$$

Low crossing results

- Among all knots with 12 or fewer crossings, it is unknown whether 37 of them are almost-alternating or have Turaev genus one (according to Knot Info).
- Our work shows that 11 of these 37 knots are not almost-alternating and do not have Turaev genus one.

Minimum crossing almost-alternating diagrams

- If $\text{adj}(u_1, u_2) \neq 1$ and $\text{adj}(v_1, v_2) \neq 1$, then D has the fewest possible crossings among almost-alternating diagrams of L.
- If $\text{adj}(u_1, u_2) = 1$ and $\text{adj}(v_1, v_2) = 1$, then L has an almost-alternating diagram with fewer crossings.
- If only one of $\text{adj}(u_1, u_2)$ or $\text{adj}(v_1, v_2)$ is equal to one, then the minimum number of crossings in an almost-alternating diagram is open.

Khovanov homology generalization

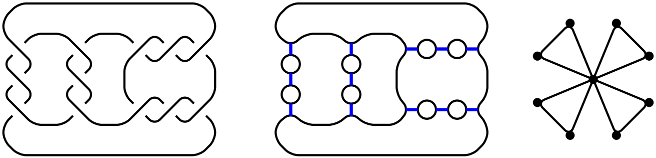
Let $j_{\min}$ and $j_{\max}$ be the least and greatest polynomial grading where the Khovanov homology of L is non-trivial.

Theorem (Dasbach, L.)

If L is almost-alternating or Turaev genus one, then either $Kh^{*,j_{\min}}(L)$ or $Kh^{*,j_{\max}}(L)$ is isomorphic to $\mathbb{Z}$.

Note. The proof relies on the long exact sequence in Khovanov homology, and so the result holds for odd Khovanov homology as well.

Semi-adequate links



D all-A state G

- A link diagram D is A-adequate if its all-A state graph G contains no loops. Similarly define B-adequate.
- A link is semi-adequate if it has a diagram that is either A-adequate or B-adequate.

Semi-adequate and almost-alternating

- Lickorish, Thistlethwaite (1988) - The leading or trailing coefficient of the Jones polynomial of a semi-adequate link is ± 1.
- Khovanov (2002) - One of the two extremal polynomial gradings of the Khovanov homology of a semi-adequate link is isomorphic to $\mathbb{Z}$.
- **Open question.** Are all almost-alternating or Turaev genus one links semi-adequate?

Happy Birthday Scott!