Khovanov homology, chromatic homology, and torsion

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Overview

- The Khovanov homology $Kh(L)$ of a link L is a categorification of the Jones polynomial of L (Khovanov - 1999).
- $Kh(L) = \bigoplus_{i,j \in \mathbb{Z}} Kh^{i,j}(L)$ is a bi-graded $\mathbb{Z}$-module.
- Experimentally, $Kh(L)$ has an abundance of torsion, only some of which we can explain.
- Among all 1,701,936 prime knots with at most 16 crossings
  1. all non-trivial knots up to 14 crossings have only 2-torsion in their Khovanov homology,
  2. 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion in their Khovanov homology, and
  3. the first known knot with odd torsion in $Kh(K)$ is the $(5,6)$-torus knot.

Motivating conjecture

Conjecture (Shumakovitch)

Let L be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.

Some more conjectures

Conjecture (Przytycki, Sazdanović - 2012)

- *The Khovanov homology of a closed 3-braid can have only 2-torsion.*
- *The Khovanov homology of a closed 4-braid can have only 2-torsion or 4-torsion.*
- *The Khovanov homology of a closed n-braid cannot have p-torsion for $p > n$, where p is prime.*

Methods

Some approaches for proving things about torsion in Khovanov homology are:

- explicit construction,
- connections with Hochschild homology,
- connections with chromatic polynomial homology, and
- spectral sequence arguments.

Computations of odd torsion

- Torus knots $(5,6)$, $(5,7)$, $(5,8)$, and $(5,9)$ have 5-torsion in their Khovanov homology.
- Przytycki and Sazdanović predicted that the closure K of

$$\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^2\sigma_1\sigma_3\sigma_2^2\sigma_4^2\sigma_3\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^3\sigma_1^2\sigma_3\sigma_2^2\sigma_4^2\sigma_3^2$$

  has 5-torsion in its Khovanov homology.
- Shumakovitch (2012) confirmed that K has 5-torsion in its Khovanov homology by showing that the difference of the Poincare polynomials of $Kh(K;\mathbb{Z}_5)$ and $Kh(K;\mathbb{Z}_7)$ is

$$(t^{12}+t^{11})q^{51}+(t^{11}+t^{10})q^{47}.$$

$Kh(K;\mathbb{Z}_5)$

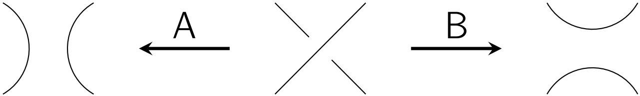
$$
\begin{aligned}
&KH_5(K) = q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2 + q^{41}t^5 + q^{43}t^5 + q^{39}t^6 + \\
&2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 + 4q^{47}t^9 + 2q^{43}t^{10} + \\
&29q^{45}t^{10} + 14q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 44q^{47}t^{11} + 31q^{49}t^{11} + q^{51}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + \\
&68q^{49}t^{12} + 42q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + \\
&159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51t^15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 9q^{57}t^{15} + 345q^{53}t^{16} + \\
&5376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + \\
&1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + \\
&9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + \\
&2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + \\
&3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + \\
&137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + \\
&759q^{75}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + \\
&218q^{81}t^{29} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + \\
&119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}
\end{aligned}
$$

$Kh(K;\mathbb{Z}_7)$

$$
\begin{aligned}
KH_7(K) \;=\;& q^{31}t^{0}+q^{33}t^{0}+q^{35}t^{2}+q^{39}t^{3}+2q^{37}t^{4}+q^{39}t^{4}+2q^{41}t^{5}+q^{43}t^{5}+q^{39}t^{6}+2q^{41}t^{6}+\\
& 2q^{43}t^{7}+2q^{45}t^{7}+4q^{41}t^{8}+3q^{43}t^{8}+q^{47}t^{8}+13q^{43}t^{9}+4q^{45}t^{9}+4q^{47}t^{9}+\\
& 2q^{43}t^{10}+29q^{45}t^{10}+13q^{47}t^{10}+q^{51}t^{10}+9q^{45}t^{11}+43q^{47}t^{11}+31q^{49}t^{11}+\\
& 2q^{45}t^{12}+34q^{47}t^{12}+68q^{49}t^{12}+41q^{51}t^{12}+2q^{53}t^{12}+11q^{47}t^{13}+85q^{49}t^{13}+\\
& 97q^{51}t^{13}+59q^{53}t^{13}+45q^{49}t^{14}+159q^{51}t^{14}+142q^{53}t^{14}+63q^{55}t^{14}+137q^{51}t^{15}+\\
& 245q^{53}t^{15}+202q^{55}t^{15}+59q^{57}t^{15}+345q^{53}t^{16}+376q^{55}t^{16}+237q^{57}t^{16}+\\
& 54q^{59}t^{16}+735q^{55}t^{17}+589q^{57}t^{17}+260q^{59}t^{17}+37q^{61}t^{17}+1328q^{57}t^{18}+\\
& 953q^{59}t^{18}+253q^{61}t^{18}+21q^{63}t^{18}+2040q^{59}t^{19}+1501q^{61}t^{19}+220q^{63}t^{19}+\\
& 9q^{65}t^{19}+2729q^{61}t^{20}+2149q^{63}t^{20}+173q^{65}t^{20}+2q^{67}t^{20}+2q^{61}t^{21}+3203q^{63}t^{21}+\\
& 2779q^{65}t^{21}+109q^{67}t^{21}+11q^{63}t^{22}+3344q^{65}t^{22}+3219q^{67}t^{22}+50q^{69}t^{22}+\\
& 36q^{65}t^{23}+3127q^{67}t^{23}+3345q^{69}t^{23}+16q^{71}t^{23}+81q^{67}t^{24}+2608q^{69}t^{24}+\\
& 3116q^{71}t^{24}+3q^{73}t^{24}+137q^{69}t^{25}+1934q^{71}t^{25}+2572q^{73}t^{25}+191q^{71}t^{26}+\\
& 1271q^{73}t^{26}+1853q^{75}t^{26}+228q^{73}t^{27}+1134q^{77}t^{27}+238q^{75}t^{28}+446q^{77}t^{28}+\\
& 568q^{79}t^{28}+219q^{77}t^{29}+294q^{79}t^{29}+218q^{81}t^{29}+759q^{75}t^{27}+175q^{79}t^{30}+226q^{81}t^{30}+\\
& 56q^{83}t^{30}+119q^{81}t^{31}+175q^{83}t^{31}+7q^{85}t^{31}+65q^{83}t^{32}+119q^{85}t^{32}+26q^{85}t^{33}+\\
& 65q^{87}t^{33}+7q^{87}t^{34}+26q^{89}t^{34}+q^{89}t^{35}+7q^{91}t^{35}+q^{93}t^{36}
\end{aligned}
$$

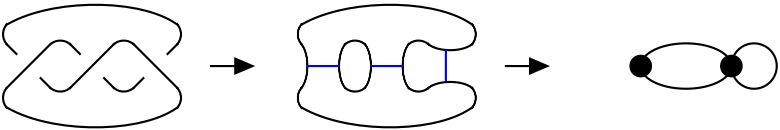
Kauffman states

- Each crossing has an A and a B resolution.



- The collection of simple closed curves in the plane obtained by taking an A or B resolution at each crossing is a *Kauffman state*.
- The all-A state graph $G_A(D)$ of D has vertices corresponding to the components of the all-A state of D and edges corresponding to the crossings of D. One can similarly define the all-B state graph $G_B(D)$.

A 3-crossing unknot



Adequate and semi-adequate links

- A link L is *adequate* if it has a diagram where both $G_A(D)$ and $G_B(D)$ have no loops.
- A link L is *semi-adequate* if it has a diagram D where either $G_A(D)$ or $G_B(D)$ has no loops.
- Alternating links are adequate.
- Many links are semi-adequate. For example, Stoimenow (2012) computed at least 249,649 of the 253,293 knots with crossing number 15 are semi-adequate.

Explicit computation results

Theorem (Asaeda, Przytycki - 2004)

1. *If $G_A(D)$ is loop-less and contains a cycle of odd length, then $Kh(D)$ contains 2-torsion.*
2. *If $G_A(D)$ is loop-less and contains a cycle of even length with an edge that is not part of a bigon, then $Kh(D)$ contains 2-torsion.*
3. *If D is prime and alternating and D is not the unknot or Hopf link, then either $G_A(D)$ or $G_B(D)$ contains an edge that is not part of a bigon. Thus $Kh(D)$ contains 2-torsion.*

Remark. Shumakovitch's conjecture is true for alternating links and "many" semi-adequate links.

Hochschild homology and $Kh(T_{2,n})$

- Let P_n be the polygon with n vertices.
- Let $C_n(A)$ be the space generated by labelings of the vertices of P_n with elements of A.
- Define a map $C_n(A) \to C_{n-1}(A)$ obtained by contracting edges and multiplying the labels on the identified vertices.
- Przytycki (2005) showed this complex gives the Hochschild homology $HH(A)$ and the Khovanov homology of $Kh(T_{2,n})$ in certain gradings.
- Allows for explicit computations of 2-torsion inside of $Kh(T_{2,n})$.

From Hochschild to chromatic polynomial cohomology

- Hochschild homology gives a sort of comultiplication free version of Khovanov homology for a polygon.
- Helme-Guizon and Rong (2004) define the chromatic polynomial cohomology. It can be simultaneously thought of as a comultiplication free version of Khovanov homology for any graph or as an extension of Hochschild homology for any graph.
- Its definition follows a similar recipe as the construction of Khovanov homology.

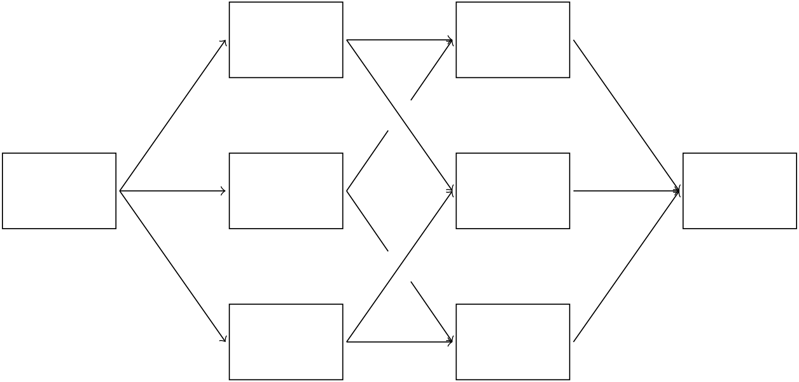
Khovanov and chromatic homology

- Let D be a link diagram, and let G be its all-A state graph.
- The Khovanov homology of D is $Kh(D)$.
- The chromatic homology of G is $H(G)$.
- Both are bigraded:

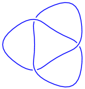
$$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D) \text{ and } H(G) = \bigoplus_{i,j} H^{i,j}(G).$$

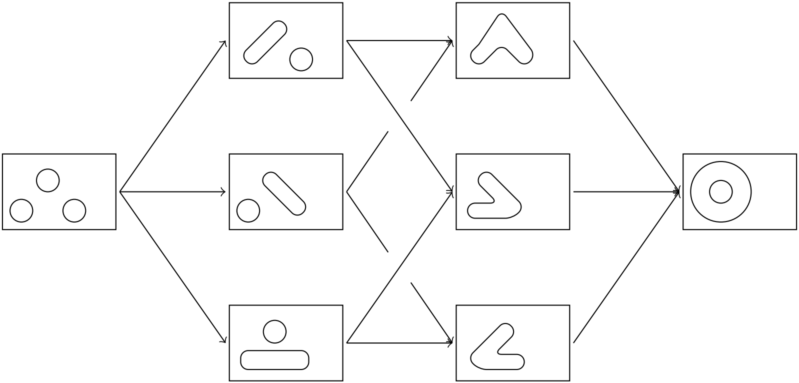
- Let $A = \mathbb{Z}[x]/(x^2)$.

Hypercube

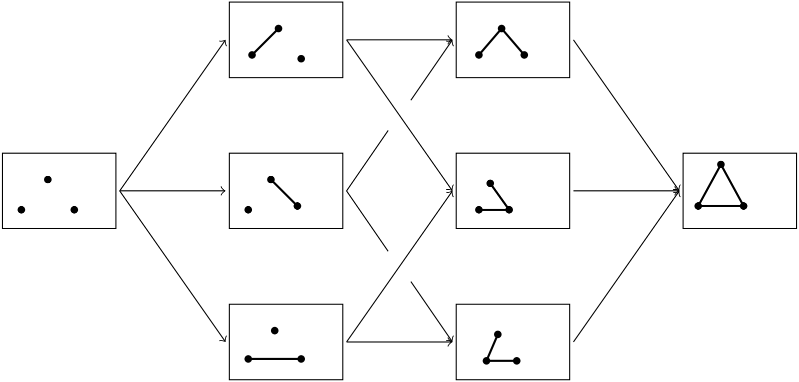


Kauffman states

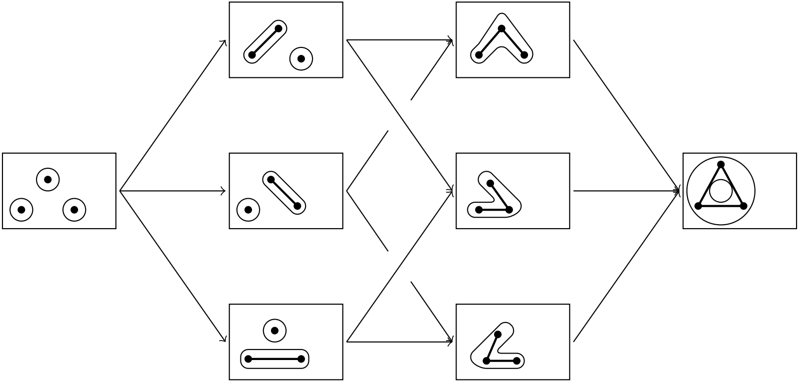




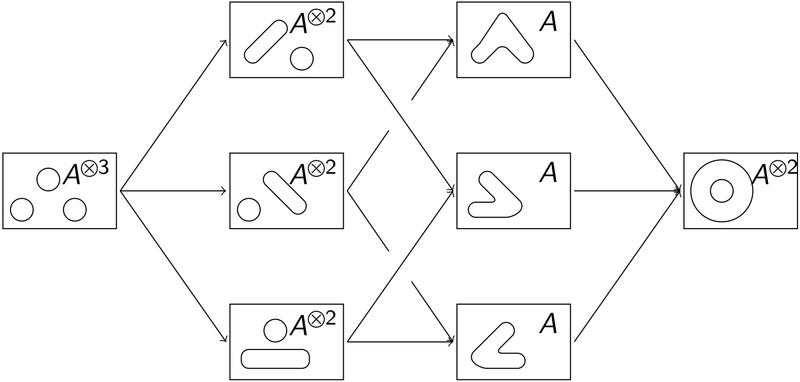
Spanning subgraphs



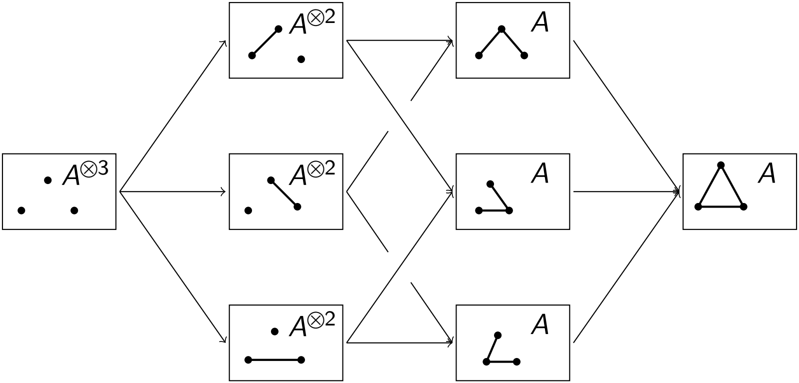
Kauffman states and spanning subgraphs



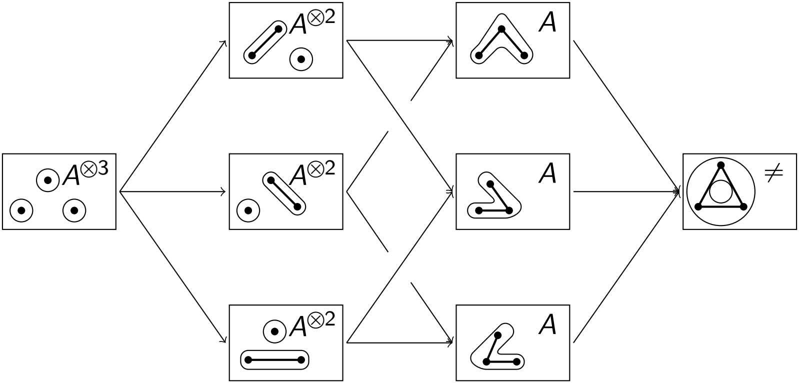
Kauffman states and spaces



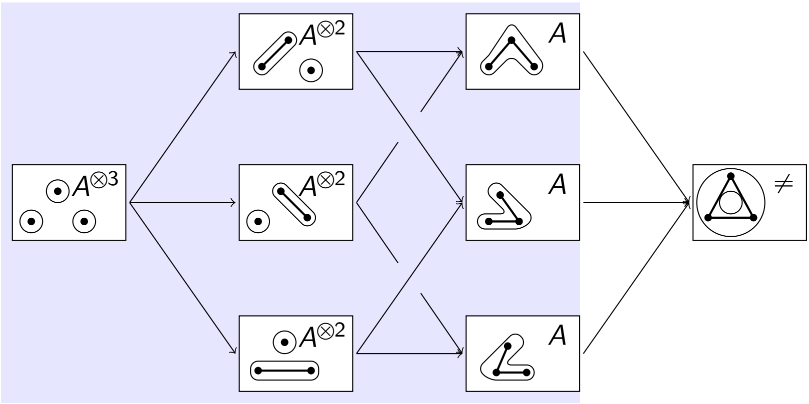
Spanning subgraphs and spaces



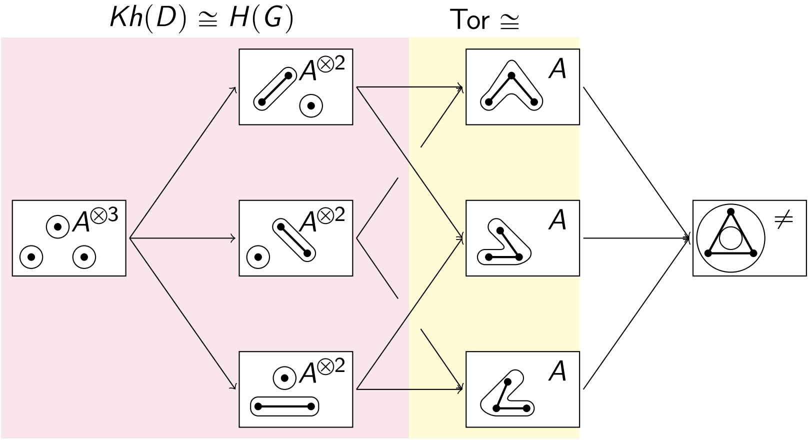
Spaces for both



Multiplication until a cycle closes



Partial Isomorphism Picture



Comparing $Kh(D)$ and $H(G)$

Theorem (Helme-Guizon, Przytycki, Rong - 2006)

If the length ℓ of the shortest cycle in G is greater than one, then there is an isomorphism between $Kh(D)$ and $H(G)$ in the first $\ell - 1$ supported i-gradings and an isomorphism of $\operatorname{Tor} Kh(D)$ *and* $\operatorname{Tor} H(G)$ *in the ℓth i-grading.*

Chromatic homology results

- Pabiniak, Przytycki, and Sazadnović (2006) use Hochschild homology and chromatic cohomology to explicitly compute Khovanov homology (including torsion) of semi-adequate links in certain gradings.
- Przytycki and Sazdanović (2012) strengthen the relationship between chromatic polynomial cohomology and Khovanov homology by modifying the comultiplication map in the chromatic complex.

More on semi-adequate links

- **So far.** The Khovanov homology of semi-adequate links where $G_A(D)$ contains an odd cycle or an even cycle with an edge that is not part of a bigon contains 2-torison.
- **New from Przytycki, Sazdanović.** The Khovanov homology of any link where $G_A(D)$ has shortest cycle with length at least 3 contains 2-torsion.
- **Result.** Shumakovitch's conjecture is true for all semi-adequate links except possibly those where $G_A(D)$ only has 2-cycles.

Our philosophy

- Chromatic homology is similar to the Khovanov homology of alternating knots.
- The previous partial isomorphism theorem says that the Khovanov homology of a link looks like the chromatic homology of a related graph in extremal homological gradings.
- Khovanov homology in extremal homological gradings should look like the Khovanov homology of an alternating knot.

Khovanov homology of alternating knots

- Lee (2002) proves that the Khovanov homology of an alternating knot is homologically thin, that is it is entirely supported on two adjacent $j - 2i$ diagonals.
- Shumakovitch (2004) proves that the Khovanov homology of an alternating knot contains no torsion of odd order.
- Shumakovitch (unpublished) proves that the Khovanov homology of an alternating knot contains only 2-torsion (and no torsion of order 2^k for $k > 1$).

Chromatic homology similarities

- Chmutov, Chmutov, and Rong (2005) prove that chromatic homology is homologically thin.
- We show that chromatic homology has no odd torsion.
- (In progress) We hope to show that chromatic homology has only 2-torsion.

The girth of a link

The *girth* of a graph G is the length of the shortest cycle in G. The *A-girth* of a link diagram D is the girth of the graph $G_A(D)$. The *A-girth* of a link L is

$$\mathrm{girth}_A(L) = \max\{\mathrm{girth}_A(D) \mid D \text{ is a diagram of } L\}.$$

One can similarly define $\mathrm{girth}_B(L)$.

Results on girth

- If K is non-trivial, then $\mathrm{girth}_A(L)$ is finite.
- Khovanov homology can provide an upper bound for $\mathrm{girth}_A(L)$. If $Kh(L)$ is homologically thick in its m-th homological grading, then $\mathrm{girth}_A(L) \leqslant m$.
- Suppose the first $m-1$ coefficients of the Jones polynomial $V_L(t)$ alternate in sign, but the $(m-1)$-st and m-th coefficients have the same sign. Then $\mathrm{girth}_A(L) \leqslant m$.

Example 9_{42}

9_{42}	-4	-3	-2	-1	0	1	2
7							1
5							1_2
3					1	1	
1				1	1_2 1		
-1				1_2 1	1		
-3		1	1				
-5		1_2					
-7	1						

$$\text{girth}_A(L) \leqslant 4$$

Spectral sequences

- Khovanov homology and related invariants arise in many spectral sequences.
- These spectral sequences are often only defined over certain coefficient rings (e.g. $\mathbb{Q}$, $\mathbb{Z}_2$, or $\mathbb{Z}_p$ for odd p).
- Use the behavior of these sequences to prove or disprove the existence of torsion.

Lee's differential

- Work over $\mathbb{Q}$ (and $\mathbb{Z}_p$ for odd p).
- Define $\mathbb{Q}$-linear maps

$$m_\Phi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m_\Phi : \begin{cases} 1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\ x \otimes 1 \mapsto 0 & x \otimes x \mapsto 1 \end{cases}$$

$$\Delta_\Phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta_\Phi : \begin{cases} 1 \mapsto 0 \\ x \mapsto 1 \otimes 1. \end{cases}$$

- Using the same conventions as in the definition of Khovanov homology, define a differential Φ on $CKh(D)$.

Lee's spectral sequence

- $(CKh(D), d, \Phi)$ form a double complex, and so there is an associated spectral sequence
- For a knot, the spectral sequence converges to $\mathbb{Q} \oplus \mathbb{Q}$.
- Shumakovitch (2004) showed that this spectral sequence also exists over $\mathbb{Z}_p$ for p an odd prime.

Gradings and Lee's spectral sequence

- Lee's differential is of bidegree $(1, 4)$.
- The bidegree of the map on the r-th page of the spectral sequence is $(1, 4r)$.
- In all known examples (over $\mathbb{Q}$) the spectral sequence collapses after the bidegree $(1, 4)$ differential.
- In such cases, $Kh(D; \mathbb{Q})$ can be arranged into "knight move" pairs.

Knight move example 7_7

	a
a	

7_7	-3	-2	-1	0	1	2	3	4
9								1
7							1	
5						2	1	
3					2	1		
1				1 + 1	2			
-1			2	1 + 2				
-3		1	1					
-5		2						
-7	1							

Analog in chromatic homology

- Chmutov, Chmutov, and Rong (2005) proved that the chromatic complex over $\mathbb{Q}$ has a differential similar to Φ.
- Leads to a spectral sequence analogous to the Lee spectral sequence.
- Chromatic homology is homologically thin and much of the homology can be arranged in "knight move" pairs.

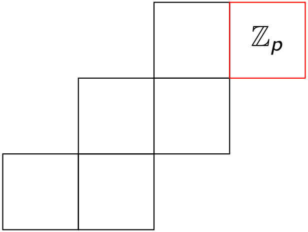
An odd torsion theorem for Khovanov homology

Theorem (Shumakovitch - 2004)

If L is homologically thin, then $Kh(L)$ contains no odd torsion.

Proof

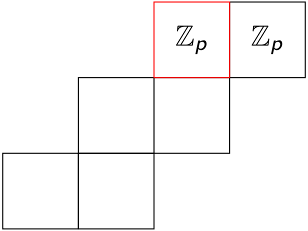
Suppose that $Kh(L)$ has p-torsion for some odd prime p. It can be shown that all torsion appears on the "bottom" diagonal. Let (i_0, j_0) be the grading with p-torsion where i_0 is minimal.



Proof

Then $\dim_{\mathbb{Z}_p} Kh^{i_0,j_0}(D) > \dim_{\mathbb{Q}} Kh^{i_0,j_0}(D)$. Since the Euler characteristic is the same over both $\mathbb{Z}_p$ and $\mathbb{Q}$,

$$\dim_{\mathbb{Z}_p} Kh^{i_0-1,j_0}(D) > \dim_{\mathbb{Q}} Kh^{i_0-1,j_0}(D).$$



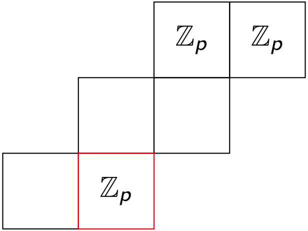
Proof

Since $Kh(D)$ lies on only two diagonals, the Lee spectral sequence implies that

$$\dim_{\mathbb{Q}} Kh^{i,j}(D) = \dim_{\mathbb{Q}} Kh^{i+1,j+4}(D) \text{ and}$$

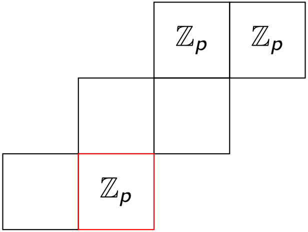
$$\dim_{\mathbb{Z}_p} Kh^{i,j}(D) = \dim_{\mathbb{Z}_p} Kh^{i+1,j+4}(D).$$

In particular, $\dim_{\mathbb{Z}_p} Kh^{i_0-2,j_0-4}(D) > \dim_{\mathbb{Q}} Kh^{i_0-2,j_0-4}(D)$.



Proof

But $\dim_{\mathbb{Z}_p} Kh^{i_0-2,j_0-4}(D) > \dim_{\mathbb{Q}} Kh^{i_0-2,j_0-4}(D)$ implies that $Kh(D)$ has p-torsion in grading $(i_0 - 2, j_0 - 4)$ which is a contradiction.



An odd torsion theorem for chromatic homology

Theorem (L., Sazdanović)

The chromatic polynomial cohomology of any graph contains no odd torsion.

Sketch of proof. Show that the Lee map in chromatic homology works over $\mathbb{Z}_p$ for odd p, and repeat the previous proof.

Application to semi-adequate links

Corollary

Let D be link diagram such that $\mathrm{girth}_A(D) > 1$*. Then* $Kh(D)$ *has no odd torsion in its first* $\mathrm{girth}_A(D)$ *supported homological gradings. Similarly, if* $\mathrm{girth}_B(D) > 1$*, then* $Kh(D)$ *has no odd torsion in its last* $\mathrm{girth}_B(D)$ *supported homological gradings.*

An odd Khovanov version

Corollary

Let D be link diagram whose all-A state graph $G_A(D)$ is planar and has girth ℓ, and let $Kh'(D)$ denote the odd Khovanov homology of the link. Then $Kh'(D)$ is torsion free in its first ℓ supported i-gradings.

Proof.

Since $G_A(D)$ is planar, there is an alternating diagram D' with $G_A(D') = G_A(D)$. One can show that $Kh'(D)$ and $Kh'(D')$ are isomorphic in the first $\ell - 1$ supported homological gradings and have isomorphic torsion in the first ℓ supported homological gradings. Because D' is alternating, its odd Khovanov homology is torsion free, and the result follows. $\square$

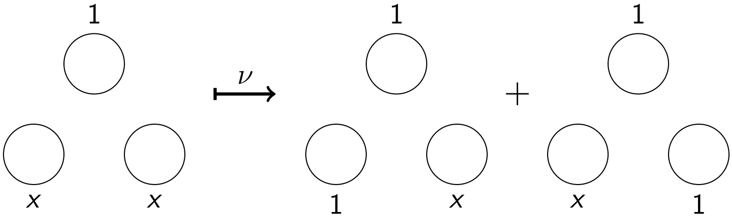
From no odd torsion to only 2-torsion

For the remainder of the talk we outline Shumakovitch's unpublished proof that the Khovanov homology of a homologically thin knot has only 2-torsion.

We end by remarking what parts of that program we have accomplished for chromatic homology.

Some maps on $Kh(D;\mathbb{Z}_2)$

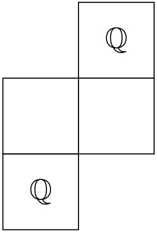
Shumakovitch defines maps ν of bidegree $(0,2)$ on $CKh(D;\mathbb{Z}_2)$.



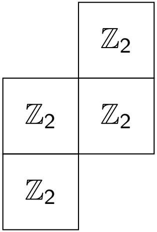
Properties of ν

- ν commutes with the Khovanov differential, and thus induces a map $\nu^* : Kh(D; \mathbb{Z}_2) \to Kh(D; \mathbb{Z}_2)$.
- Homology with respect to ν is trivial, and so ν^* is an isomorphism.
- The "vertical" Euler characteristic of $Kh(D; \mathbb{Z}_2)$ is trivial.

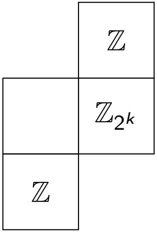
Knight moves - what we've shown so far



over $\mathbb{Q}$

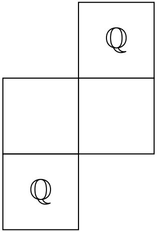


over $\mathbb{Z}_2$

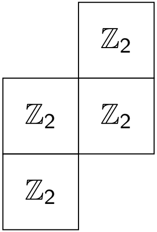


over $\mathbb{Z}$

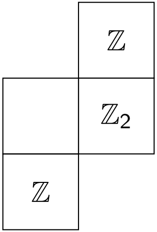
Knight moves - what we will show



over $\mathbb{Q}$



over $\mathbb{Z}_2$



over $\mathbb{Z}$

Spectral sequence associated to an exact couple

$$\begin{tikzcd}
D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\
& E_1 \arrow[ul, "k_1"] &
\end{tikzcd}$$

Given the above exact triangle, define $d_1 = j_1 \circ k_1 : E_1 \to E_1$. Then $d_1^2 = j_1 \circ k_1 \circ j_1 \circ k_1 = 0$. Define $E_2 = H(E_1, d_1)$ and $D_2 = \mathrm{Im}(i_1) = \mathrm{Ker}(j_1)$. Then maps i_2, j_2, and k_2 can be defined so that the following triangle is exact.

$$\begin{tikzcd}
D_2 \arrow[rr, "i_2"] & & D_2 \arrow[dl, "j_2"] \\
& E_2 \arrow[ul, "k_2"] &
\end{tikzcd}$$

Iterating this process yields a spectral sequence $\{E_r, d_r\}$.

A long exact sequence

The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$

induces a short exact sequence of complexes

$$0 \to CKh(D) \xrightarrow{\times 2} CKh(D) \xrightarrow{\text{mod } 2} CKh(D; \mathbb{Z}_2) \to 0,$$

which induces a long exact sequence on homology

$$\cdots \to Kh(D) \xrightarrow{\times 2} Kh(D) \xrightarrow{\text{mod } 2} Kh(D; \mathbb{Z}_2) \xrightarrow{\partial} Kh(D) \to \cdots .$$

Bockstein spectral sequence

$$\begin{tikzcd} Kh(D) \arrow[rr, "\times p"] & & Kh(D) \arrow[dl, "\bmod p"] \\ & Kh(D;\mathbb{Z}_2) \arrow[ul, "\partial"] \arrow[loop below, "\beta"] & \end{tikzcd}$$

The above triangle is an exact couple. Define $\beta : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$ by $\beta = \partial \mod p$.

There exists a spectral sequence $\{B_r, b_r\}$ with $B_1 = Kh(D;\mathbb{Z}_2)$ and $b_1 = \beta$ that converges to the free part of $Kh(D)$ tensored with $\mathbb{Z}_2$.

More on the Bockstein spectral sequence

Theorem

$Kh(D)$ has no 2^k torsion if and only if the Bockstein spectral sequence collapses at the kth page.

New Goal. Show that the Bockstein spectral sequence collapses at the first page for homologically thin links.

Turner's differential with $\mathbb{Z}_2$ coefficients

- Define $\mathbb{Z}_2$-linear maps

$$m_T : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m_T : \begin{cases} 1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\ x \otimes 1 \mapsto 0 & x \otimes x \mapsto x \end{cases}$$

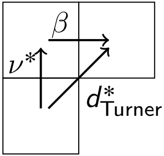
$$\Delta_T : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta_T : \begin{cases} 1 \mapsto 1 \otimes 1 \\ x \mapsto 0. \end{cases}$$

- Using the same conventions as in the definition of Khovanov homology, define a differential d_{Turner} on $CKh(D; \mathbb{Z}_2)$.

Turner's spectral sequence on $Kh(D;\mathbb{Z}_2)$

- $(CKh(D;\mathbb{Z}_2), d, d_{\text{Turner}})$ form a double complex, and so there is an associated spectral sequence.
- d_{Turner} commutes with the usual Khovanov differential, and so there is an induced map $d^*_{\text{Turner}} : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$.
- For a knot, the above spectral sequence converges to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If the homology is thin, the last non-zero map in the spectral sequence is d^*_{Turner}.

Putting it all together



1. $\nu^* : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$ is an isomorphism.
2. The Turner spectral sequence collapses at the first page.
3. $d^*_{\text{Turner}} = \nu^* \circ \beta + \beta \circ \nu^*$.
4. (1) - (3) implies that the Bockstein spectral sequence collapses after the first page.

Torsion in Khovanov homology of homologically thin knots

Theorem (Shumakovitch)

If K is homologically thin, then its Khovanov homology only has 2-torsion.

Corollary

The Khovanov homology of an alternating knot is determined by its Jones polynomial.

Corollary

If K is a nontrivial, homologically thin knot, then its Khovanov homology contains 2-torsion.

Remark. The second corollary requires Kronheimer-Mrowka's result that Khovanov homology detects the unknot.

Our progress

1. Maps analogous to ν and d_{Turner} exist in chromatic homology and commute with the chromatic differential.
2. The map ν^* is an isomorphism on chromatic homology.
3. The chromatic Turner spectral sequence collapses where it is supposed to.
4. The Bockstein spectral sequence can be defined on chromatic homology.
5. **Remains to show.** $d^*_{\text{Turner}} = \nu^* \circ \beta + \beta \circ \nu^*$.

Consequences if (5) is true

- Chromatic homology only contains 2-torsion.
- Khovanov homology only contains 2-torsion in the first $\mathrm{girth}_A(D)$ and last $\mathrm{girth}_B(D)$ homological gradings.
- Chromatic homology is determined by the chromatic polynomial (not true when working with other algebras).

Thank you!