The Jones polynomial of almost alternating and Turaev genus one links

Adam Lowrance - Vassar College
Oliver Dasbach - LSU

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Almost alternating links (Adams et. al.)

A non-alternating link is *almost alternating* if it has a diagram that can be transformed into an alternating diagram via one crossing change.

A prime almost alternating knot is either hyperbolic, $T(3,4)$ or $T(3,5)$ (Abe, Kishimoto).

All non-alternating knots with eleven or fewer crossings except possibly $11n_{95}$ and $11n_{118}$ are almost-alternating.

Not almost alternating?

A non-alternating knot K is not almost alternating if

- K is prime and satellite,
- the reduced Khovanov homology $\widetilde{Kh}(K)$ lies on more than two adjacent diagonals (Wehrli, Champanerkar-Kofman), or
- the knot Floer homology $\widehat{HFK}(K)$ lies on more than two adjacent diagonals (Ozsváth, Szabó).

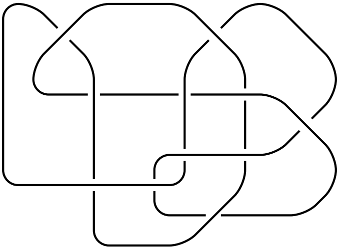
The Jones polynomial of an almost alternating link

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

Theorem (Dasbach, L.)

If L is almost-alternating, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

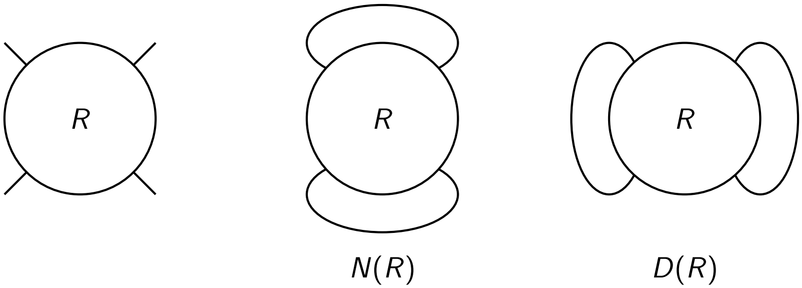
A low crossing result



$11n_{95}$

$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9$ implies $11n_{95}$ is not almost alternating.

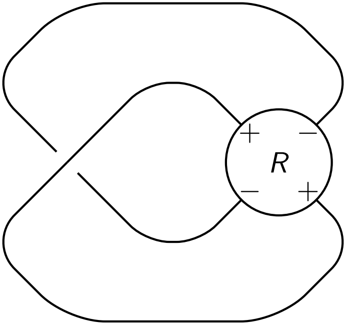
Sketch: tangle closures



A tangle R, its numerator closure $N(R)$ and its denominator closure $D(R)$.

Sketch: almost alternating diagram

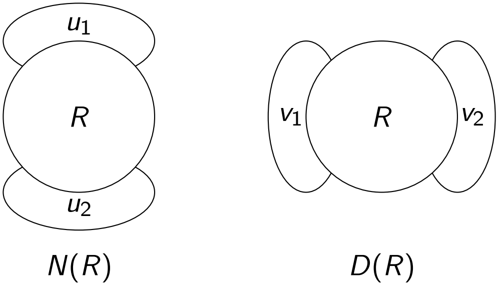
If D is almost-alternating, then it has a diagram as below.



Sketch: Kauffman bracket computation

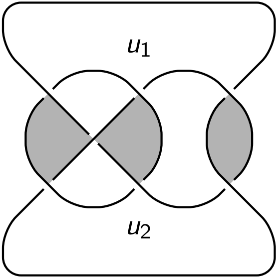
- $\langle D \rangle = A\langle D(R) \rangle + A^{-1}\langle N(R) \rangle$.
- Both $D(R)$ and $N(R)$ are alternating diagrams.
- For any alternating diagram D_{alt}, the extreme coefficients of $\langle D_{\text{alt}} \rangle$ are ± 1 (Kauffman).
- The extreme coefficients of $A\langle D(R) \rangle$ and $A^{-1}\langle N(R) \rangle$ cancel with one another.

Sketch: adjacent faces

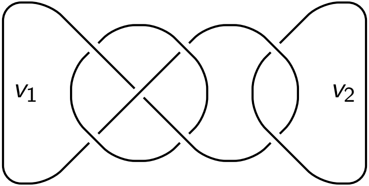


- Let $\mathrm{adj}(u_1, u_2)$ be the number of faces of $N(R)$ adjacent to both u_1 and u_2.
- Let $\mathrm{adj}(v_1, v_2)$ be the number of faces of $D(R)$ adjacent to both v_1 and v_2.

Sketch: adjacent faces example



$\text{adj}(u_1, u_2) = 3$



$\text{adj}(v_1, v_2) = 0$

Sketch: more Kauffman bracket

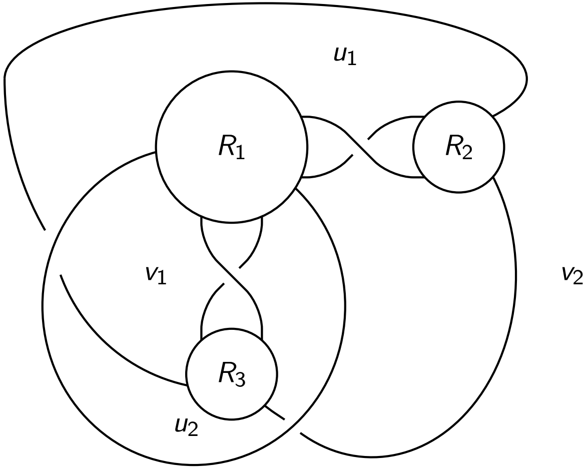
- Dasbach, Lin (2006) implies that the absolute values of the extreme terms of $\langle D \rangle$ are

$$|\operatorname{adj}(u_1, u_2) - 1| \text{ and } |\operatorname{adj}(v_1, v_2) - 1|.$$

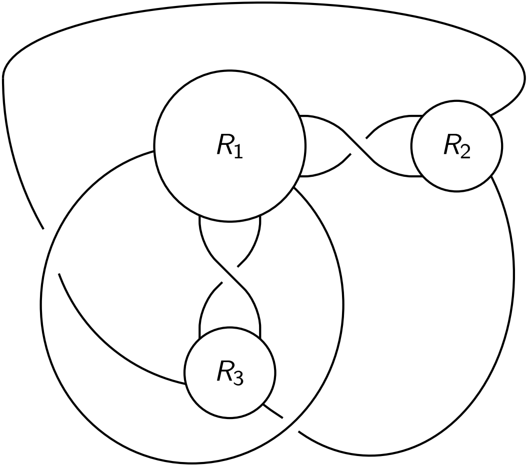
- If $\operatorname{adj}(u_1, u_2) \geq 3$, then $\operatorname{adj}(v_1, v_2) = 0$ (and vice versa).
- Theorem holds unless $\operatorname{adj}(u_1, u_2) = \operatorname{adj}(v_1, v_2) = 1$.
- If $\operatorname{adj}(u_1, u_2) = \operatorname{adj}(v_1, v_2) = 1$, then L has an almost-alternating diagram with fewer crossings than D.

Sketch: $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$

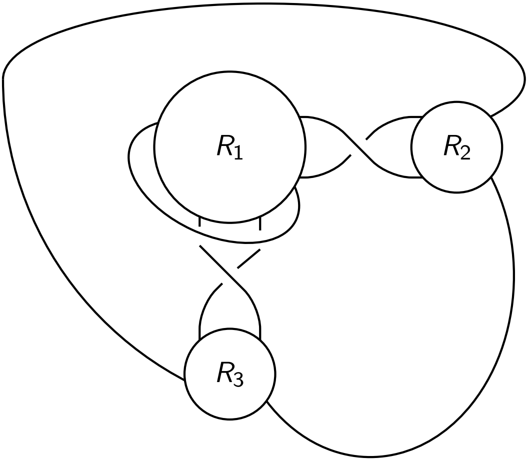
If $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$, then D has diagram as below.



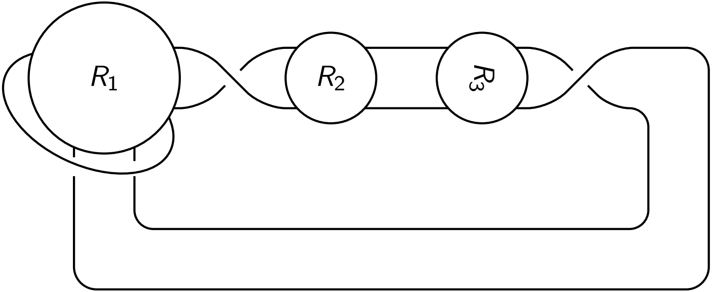
Sketch: isotopy



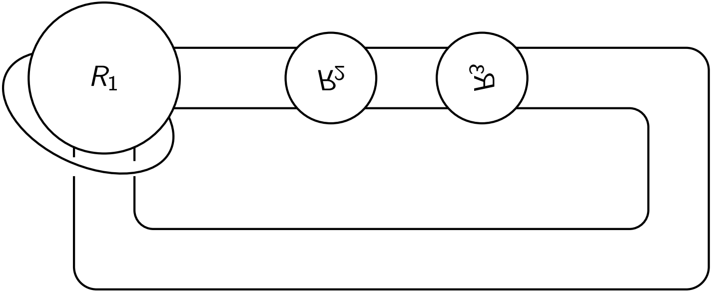
Sketch: isotopy



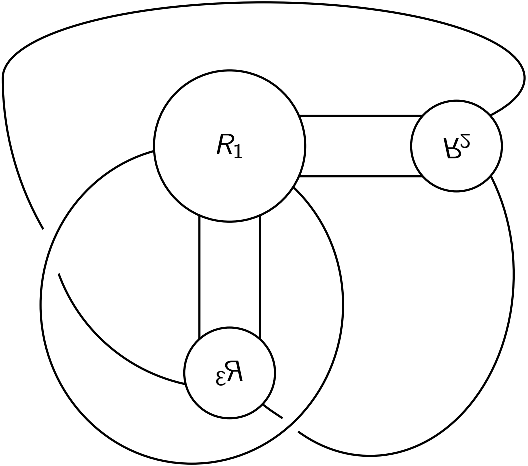
Sketch: isotopy



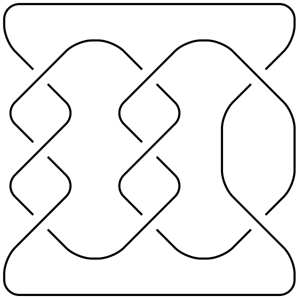
Sketch: isotopy

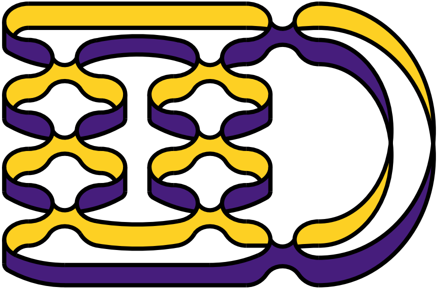


Sketch: isotopy



The Turaev surface in pictures





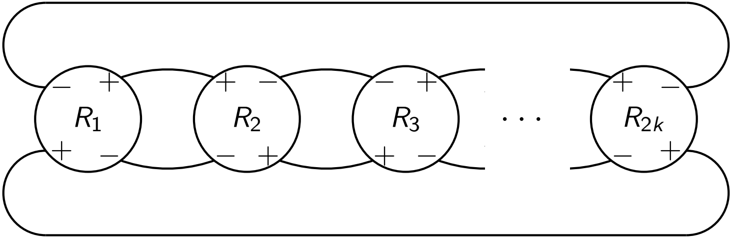
Turaev genus

For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface $F(D)$.

The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Classification of links of Turaev genus one



Theorem (Armond, L.; Kim)

1. *Let* $R_1, \ldots, R_{2k}$ *be alternating two-tangles, and let* D *be a link diagram connecting* $R_1, \ldots, R_{2k}$ *as depicted above. Then* $g_T(D) = 1$.
2. *Moreover, if* L *is a non-split link with* $g_T(L) = 1$*, then* L *has a diagram as above.*

Mutation

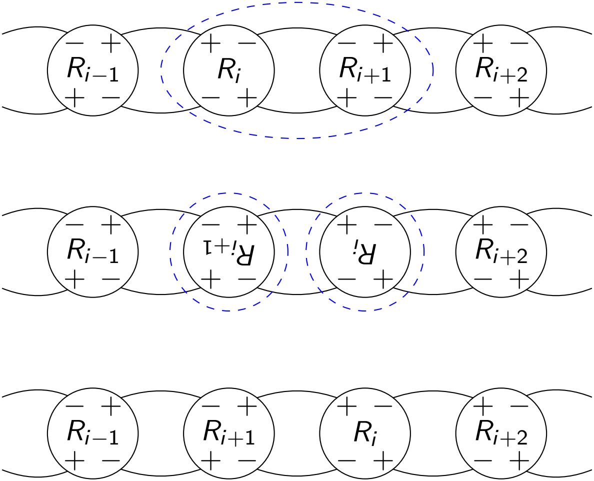
Theorem (Armond, L.)

If $g_T(L) = 1$, then L is mutant to an almost-alternating link.

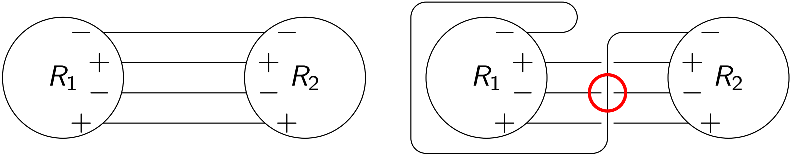
Corollary

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero. If L is Turaev genus one, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Mutation proof



Mutation proof continued



Questions

- Does $g_T(L) = 1$ imply almost alternating?
- Does almost alternating imply semi-adequate?

Thank you!