Computing the Turaev genus of a link

Adam Lowrance - Vassar College

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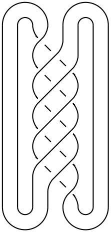
Construction of the Turaev surface $F(D)$

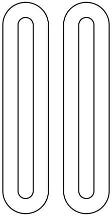
1. Replace arcs of D not near crossings with bands transverse to the projection plane.
2. Replace crossings of D with saddles interpolating between the all-B and all-A states of D
3. Cap off the boundary components with disks to obtain $F(D)$.

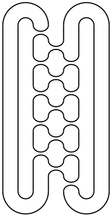
The Turaev surface in pictures



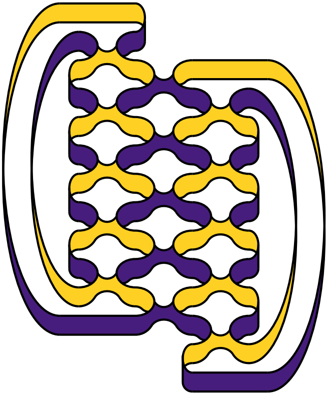
D

all-A state





all-B state



$F(D)$

Turaev genus

- For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface $F(D)$.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Some history of the Turaev surface

- Turaev (1987) related $F(D)$ to the difference between the span of the Jones polynomial and the crossing number.
- DFKLS (2006) - The Jones polynomial is an evaluation of the Bollobás-Riordan-Tutte polynomial of a graph embedded on the Turaev surface.
- CKS (2007), DL (2007, 2009, 2015) - Connections with Khovanov homology and knot Floer homology.
- Kalfagianni (2016) - Characterization of adequate knots using the colored Jones polynomial and Turaev genus.

Turaev genus of torus knots

- (Turaev) A link is alternating if and only if $g_T(L) = 0$.
- Since $T_{2,q}$ is alternating, we have $g_T(T_{2,q}) = 0$ for all q.
- (Abe, Kishimoto) The Turaev genus of a three stranded torus knot is given by
$$g_T(T_{3,3n+i}) = n$$
where $i = 1$ or 2 and n is a nonnegative integer.

Turaev genus of torus knots

Theorem (Jin, L., Polston, Zheng)

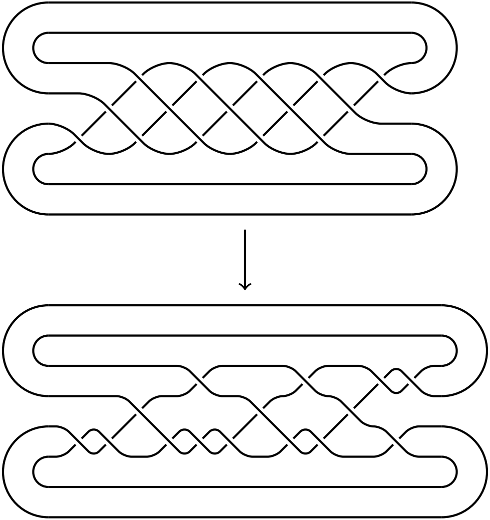
For each nonnegative integer n, we have

$$g_T(T_{4,4n+1}) = 2n,$$
$$g_T(T_{4,4n+3}) = 2n+1,$$
$$g_T(T_{5,5n+1}) = 4n, \text{ and}$$
$$g_T(T_{6,6n+1}) = 6n.$$

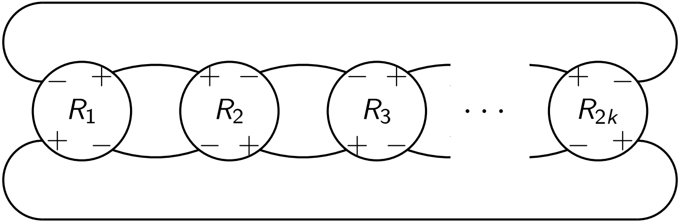
Sketch of proof

- (Moser) Surgery with coefficient $pq \pm 1$ on $T_{p,q}$ yields a lens space.
- (Ozsváth, Szabó) If K has a lens space surgery, then there is an algorithm for computing $\widehat{HFK}(K)$ from $\Delta_K(t)$.
- (L.) The width of $\widehat{HFK}(K)$ gives a lower bound on $g_T(K)$.
- Find diagrams of $T_{p,q}$ that realize the lower bound.

Turaev genus minimizing diagram: $T_{4,5}$



Classification of links of Turaev genus one



Theorem (Armond, L.; Kim)

1. *Let $R_1, \ldots, R_{2k}$ be alternating two-tangles, and let D be a link diagram connecting $R_1, \ldots, R_{2k}$ as depicted above. Then $g_T(D) = 1$.*
2. *Moreover, if L is a non-split link with $g_T(L) = 1$, then L has a diagram as above.*

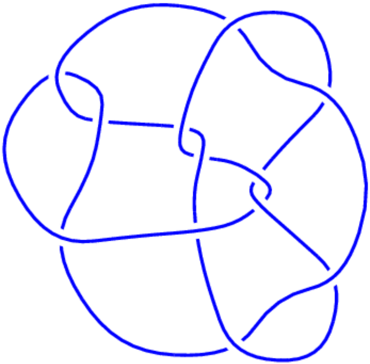
Jones polynomial results

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

Theorem (Dasbach, L.)

If $g_T(L) = 1$, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

An example: $12n375$



$$V_{12n375}(t) = 2t^2 - 4t^3 + 8t^4 - 9t^5 + 10t^6 - 10t^7 + 7t^8 - 5t^9 + 2t^{10}$$

Low crossing results

- Among all knots with 12 or fewer crossings, it was unknown whether 37 of them are Turaev genus one or two (according to Knot Info).

- Our work shows that 11 of these 37 knots are Turaev genus two.

Thank you!