Khovanov homology, chromatic homology, and torsion

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The chromatic polynomial

- Let G be a graph and let t be a positive integer. Define $p_G(t)$ to be the number of proper colorings of the vertex set of G using the colors $1, 2, \ldots, t$.
- The function $p_G(t)$ is a polynomial in the variable t and is called the *chromatic polynomial* of G. It was defined by Birkhoff in 1912.
- The chromatic polynomial can be expressed as a state sum.
- The chromatic polynomial of is an evaluation of the Tutte polynomial $T_G(x, y)$.

The Jones polynomial

- The Jones polynomial $V_L(t)$ was defined by Jones in 1984.
- The Jones polynomial has a state sum formulation (due to Kauffman).
- The Jones polynomial of an alternating link L is a specialization of the Tutte polynomial of its checkerboard graph G.

Homological invariants

- In 1999, Khovanov constructed Khovanov homology $Kh(L)$, a categorification of the Jones polynomial.
- In 2004, Helme-Guizon and Rong constructed the chromatic polynomial homology $H(G)$, a categorification of the chromatic polynomial.
- Helme-Guizon, Przytycki, and Rong (2005) and Pabiniak, Przytycki, and Sazdanović (2006) show that there is a partial isomorphism between $Kh(L)$ and $H(G)$.

Guiding principle

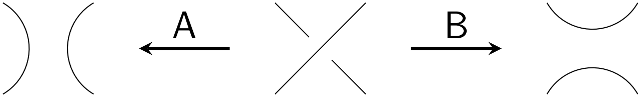
- The Khovanov homology of an alternating link is well-understood. Lee (2002) showed $Kh(L)$ is homologically thin, and Shumakovitch (2004 and 201?) showed that $Kh(L)$ has only 2-torsion.
- Chromatic polynomial homology is similar to the Khovanov homology of an alternating link.
- Use intuition about the Khovanov homology of alternating links to prove statements about chromatic polynomial homology. Then use the partial isomorphism to prove new results about Khovanov homology of certain non-alternating links.

A connection between link diagrams and graphs

Link diagram $D \rightsquigarrow$ all-A state graph G

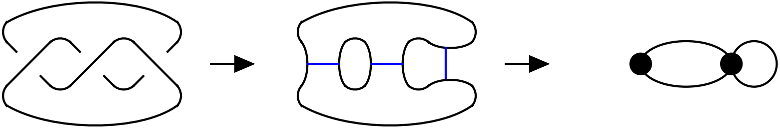
The all-A state graph

- Each crossing has an A and a B resolution.



- The collection of simple closed curves in the plane obtained by taking an A or B resolution at each crossing is a *Kauffman state*.
- The all-A state graph G of D has vertices corresponding to the components of the all-A state of D and edges corresponding to the crossings of D.

A 3-crossing unknot



D G

The partial isomorphism

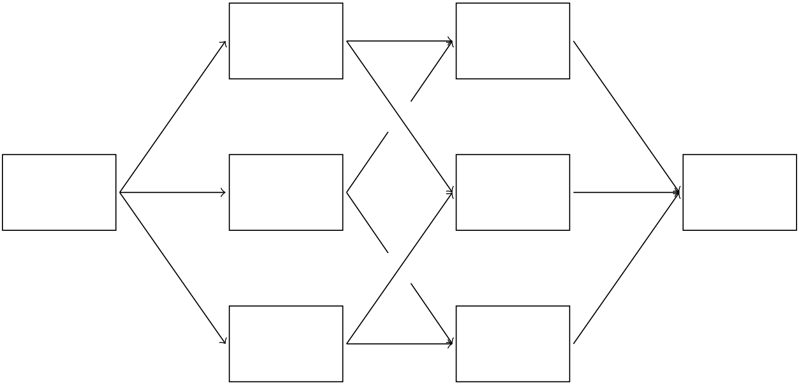
- Both Khovanov homology and chromatic polynomial homology are bi-graded:

$$Kh(L) = \bigoplus Kh^{i,j}(L) \text{ and } H(G) = \bigoplus H^{i,j}(G).$$

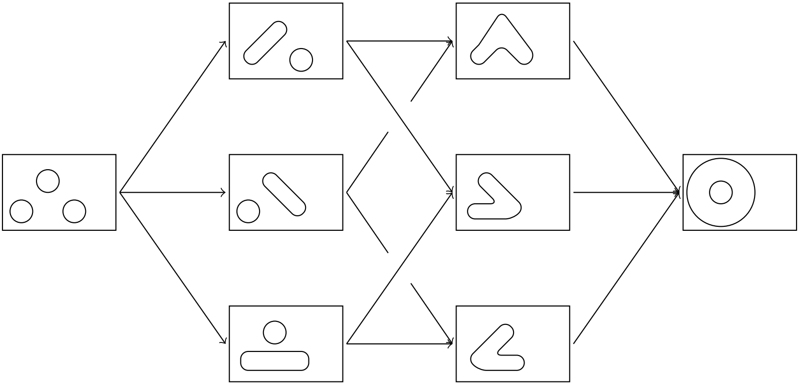
- There is an isomorphism between $Kh^{i,j}(L)$ and $H^{\tilde{i},\tilde{j}}(G)$ for certain values of $i, j, \tilde{i}$, and $\tilde{j}$. It arises by comparing the cube of resolution constructions for each theory.

- Let A denote the algebra $\mathbb{Z}[x]/(x^2)$.

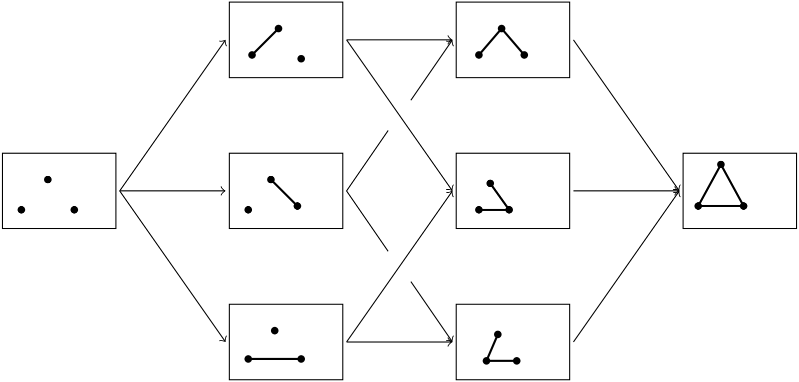
Hypercube - both $Kh(L)$ and $H(G)$



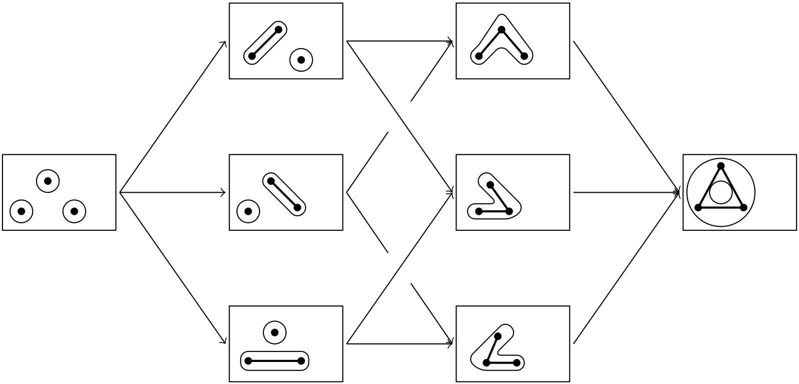
Kauffman states - $Kh(L)$



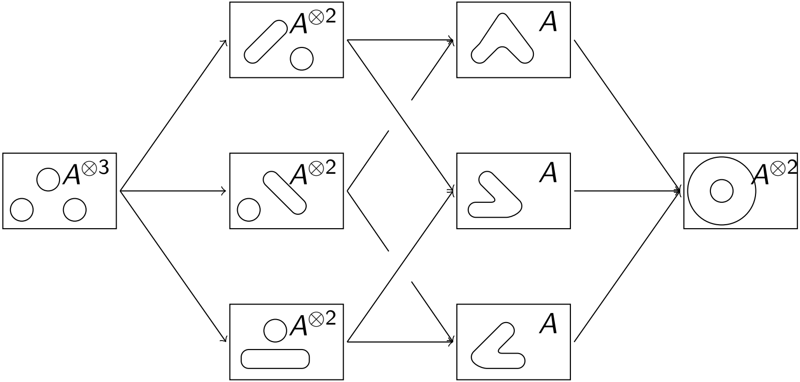
Spanning subgraphs - $H(G)$



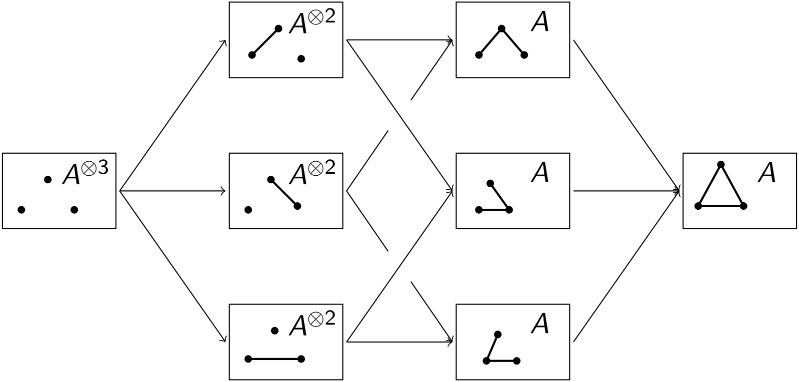
Kauffman states and spanning subgraphs



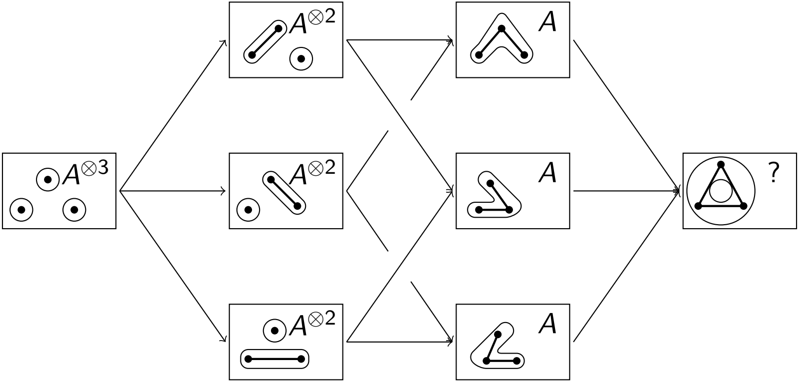
Kauffman states and spaces - $Kh(L)$



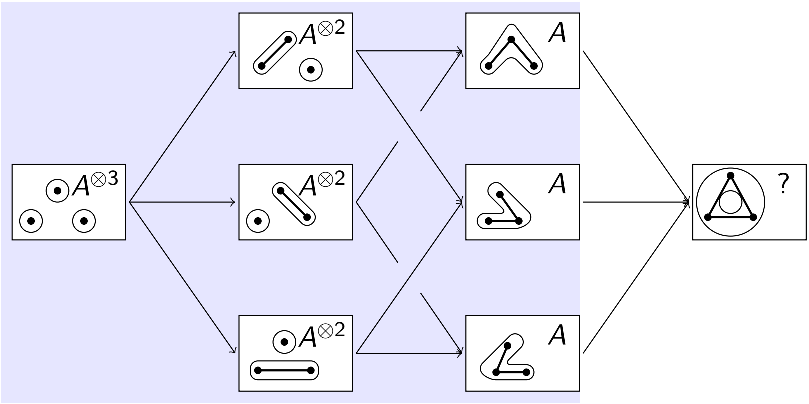
Spanning subgraphs and spaces - $H(G)$



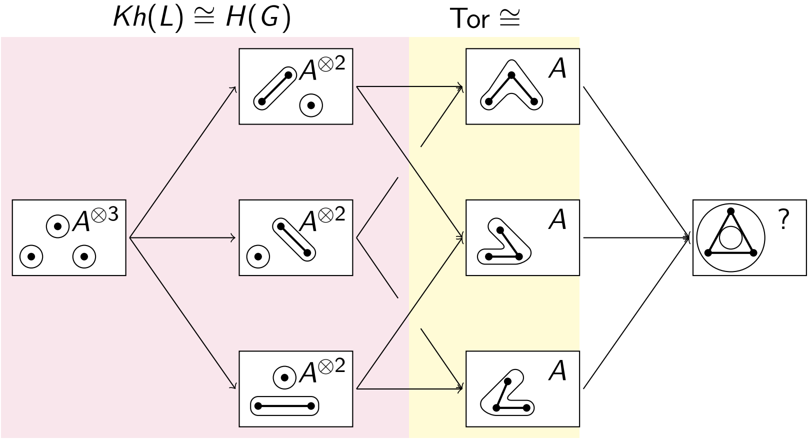
Spaces for both



Multiplication until a cycle closes



Partial Isomorphism Picture



Statement of partial isomorphism theorem

Theorem (HGPR and PPS)

Let D be a diagram of the link L with all-A state graph G and n_- negative crossings. Suppose that the length of the shortest cycle in G is ℓ. Then

$$Kh^{i,*}(L) \cong H^{i+n_-,*}(G)$$

whenever $0 \leq i < \ell - 1$. Moreover,

$$\mathrm{Tor}\, Kh^{\ell-1,*}(L) \cong \mathrm{Tor}\, H^{\ell-1+n_-,*}(G).$$

Torsion in $Kh(L)$

- Experimentally, $Kh(L)$ has an abundance of torsion, only some of which we can explain.
- In almost all examples, $Kh(L)$ contains 2-torsion.
- Torsion of order other than 2 appears, but is much more rare.

Motivating conjecture

Conjecture (Shumakovitch)

Let L be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.

Some more conjectures

Conjecture (Przytycki - Sazdanović)

- *The Khovanov homology of a closed 3-braid can have only 2-torsion.*
- *The Khovanov homology of a closed 4-braid can have only 2-torsion or 4-torsion.*
- *The Khovanov homology of a closed n-braid cannot have p-torsion for $p > n$, where p is prime.*

Following our guiding principle - torsion

- (Shumakovitch - 2004) The Khovanov homology of an alternating link does not contain odd torsion.
- (L., Sazdanović - 2015) Chromatic polynomial homology does not contain odd torsion.
- (L., Sazdanović - 2015) If D is a diagram of L and its all-A state graph G has shortest cycle of length ℓ, then $Kh^{i,*}(L)$ contains no odd torsion when $-n_{-} \leq i \leq -n_{-} + \ell - 1$.

A similar theorem for odd Khovanov homology

Odd Khovanov homology $Kh_{\text{odd}}(L)$ was developed by Ozsváth, Rasmussen, and Szabó in 2009. If L is alternating, then $Kh_{\text{odd}}(L)$ is torsion-free.

Theorem (L., Sazdanović)

Let D be a diagram of L whose all-A state graph is planar and has shortest cycle of length ℓ. Then $Kh^{i,*}_{\text{odd}}(L)$ is torsion-free when $-n_- \leq i \leq -n_- + \ell - 1$.

Ongoing work on torsion

- (Shumakovitch - 201?) The Khovanov homology of an alternating link has only 2-torsion.
- **Question.** Does chromatic polynomial homology only contain 2-torsion?
- If so, we can use the partial isomorphism theorem to make an analogous statement about Khovanov homology.

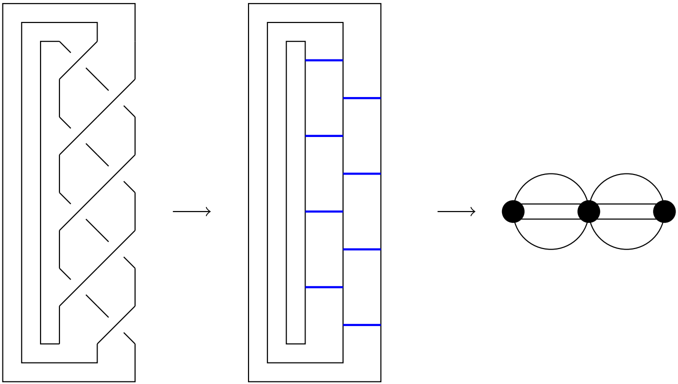
Following our guiding principles - homological thickness

- (Lee 2002) If L is a non-split alternating link, then $Kh^{i,j}(L)$ is entirely supported on two adjacent $2i - j$ diagonals.
- (Chmutov, Chmutov, Rong 2005) Chromatic polynomial homology is entirely supported on two adjacent $i + j$ diagonals.
- If D is a diagram of L whose all-A state graph has shortest cycle of length ℓ, then $Kh^{i,j}(L)$ is entirely supported on two adjacent $2i - j$ diagonals whenever $i < \ell - 1 - n_-$.

Girth

- Let G be the all-A state graph of a diagram D. Define the girth of D to be the length of the shortest cycle in G.
- Define the girth of L to be the maximum girth of D where D is a diagram of L. Denote it $\text{girth}(L)$.

Ribbon girth example



$$\mathrm{girth}(D) = 2$$

A Khovanov homology bound on girth

- Let i_0 be the least i-grading in which $Kh(L)$ is supported. If D is a diagram of L such that G has no loops, then $i_0 = -n_-$.
- Let i_{thick} be the least i-grading where $Kh^{i,j}(L)$ is not entirely supported on two adjacent $2i - j$ diagonals.
- Then $\text{girth}(L) \leq i_{\text{thick}} - i_0 + 1$.

Example 9_{42}

9_{42}	-4	-3	-2	-1	0	1	2
7							1
5							1_2
3					1	1	
1				1	1_2 1		
-1				1_2 1	1		
-3		1	1				
-5		1_2					
-7	1						

$$i_0 = -4, i_{\text{thick}} = 0$$

$$\text{girth}(L) \leq 5$$

Rasmussen's invariant

- Let $s(K)$ be the Rasmussen concordance invariant of K.
- Let D be a diagram of K with n_+ positive crossings, n_- negative crossings, and whose all-A state graph has v vertices.
- (Rasmussen - 2004) If $n_- = 0$ (i.e. D is a positive diagram), then $s(K) = -v + n_+ + 1$.
- (Tagami - 2014) Computes $s(K)$ in the case where $n_- = 1$.

A Rasmussen invariant computation

Theorem (L., Sazdanović)

Let D be a diagram of a knot K with n_+ positive crossings, n_- negative crossings. Suppose that the all-A state graph G of D has v vertices and shortest cycle of length ℓ. If $n_- < \ell - 1$, then

$$s(K) = -v + n_+ + 1.$$

Question about genus

If $n_- = 0$ or 1, Rasmussen and Tagami respectively showed that

$$g_3(K) = g_4(K) = \frac{1}{2}s(K).$$

Question. Is $g_3(K) = g_4(K) = \frac{1}{2}s(K)$ whenever $n_- < \ell - 1$?

Thank you!