Invariants of Turaev genus one knots

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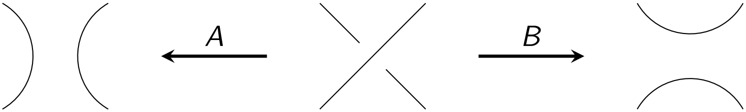
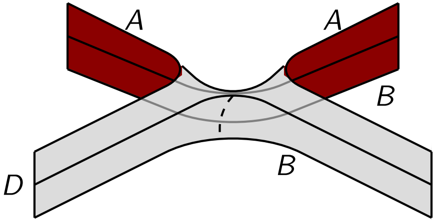
Construction of the Turaev surface $F(D)$

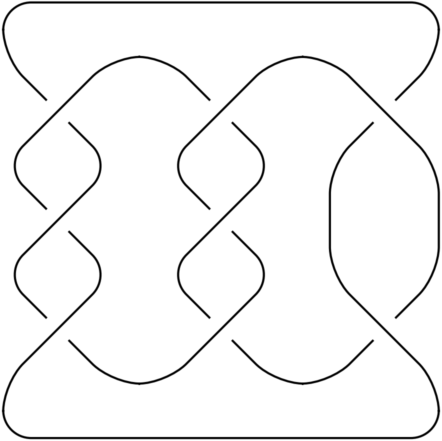
1. Replace arcs of D not near crossings with bands transverse to the projection plane.
2. Replace crossings of D with saddles interpolating between the all-B and all-A states of D
3. Cap off the boundary components with disks to obtain $F(D)$.

The Turaev surface in pictures

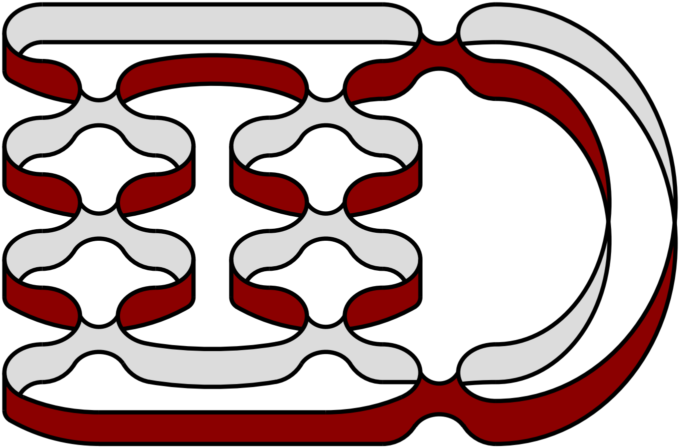




The pretzel knot $P(3, 3, -2)$



The Turaev surface $F(P(3, 3, -2))$



Turaev genus

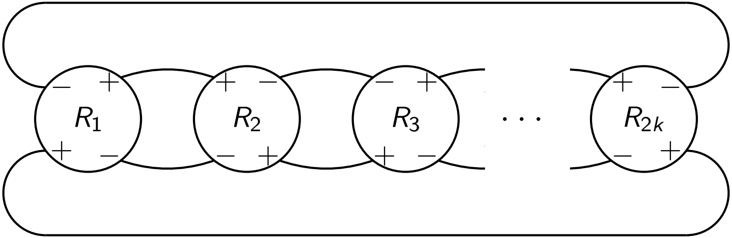
- For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface $F(D)$.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

Some history of the Turaev surface

- Turaev (1987) related $F(D)$ to the difference between the span of the Jones polynomial and the crossing number.
- DFKLS (2006) - Connections with the Jones polynomial.
- Champanerkar, Kofman, and Stoltzfus (2007) - Connections with spanning tree version of Khovanov homology.
- L. (2007) - Connections with spanning tree version knot Floer homology.
- Dasbach, L. (2009) - Connections with signature, and the s and τ invariants.
- Dasbach, L. (2015) - A Turaev surface model of Khovanov homology.

Classification of links of Turaev genus one

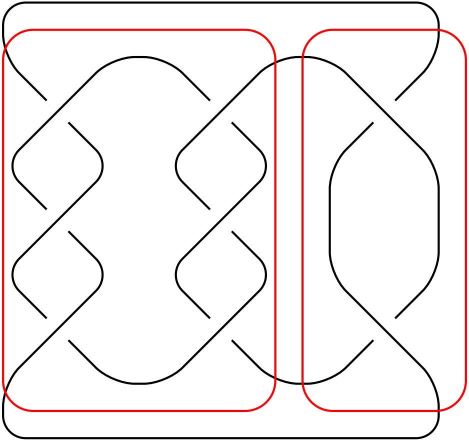


Theorem (Armond, L.; Kim)

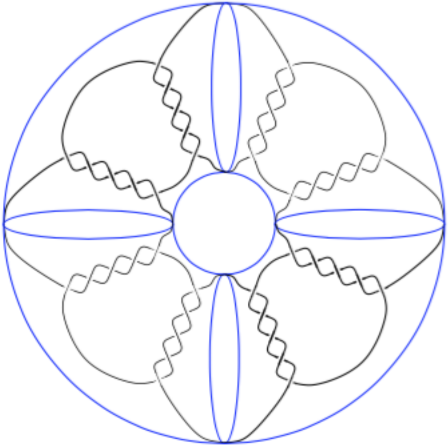
1. *Let $R_1, \ldots, R_{2k}$ be alternating two-tangles, and let D be a link diagram connecting $R_1, \ldots, R_{2k}$ as depicted above. Then $g_T(D) = 1$.*
2. *Moreover, if L is a non-split link with $g_T(L) = 1$, then L has a diagram as above.*

Examples: Pretzel links

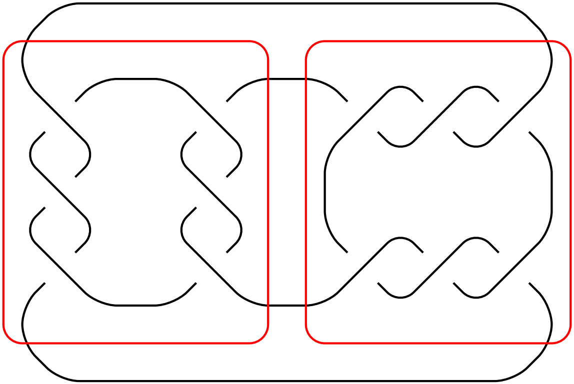
If L is a non-alternating pretzel link, then $g_T(L) = 1$.



A pretzel link on the torus



Another example

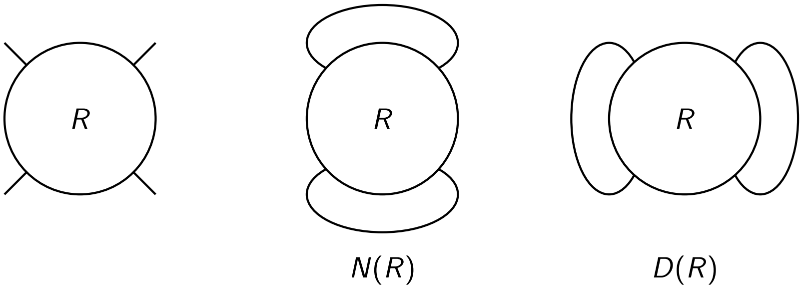


$$g_T(K) = 1$$

Determinant and signature of a knot

- Let V be a Seifert matrix for the knot K.
- The determinant is $\det(K) = |\det(V + V^T)|$.
- The signature of is $\sigma(K) = \sigma(V + V^T)$.

Numerator and denominator



A tangle R, its numerator closure $N(R)$ and its denominator closure $D(R)$.

A determinant formula

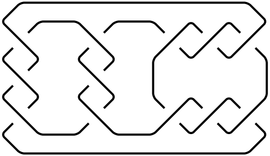
Theorem (Dasbach, L.)

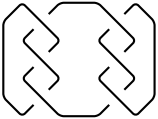
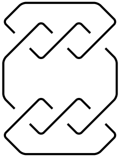
Let L be a link with diagram D as in the Turaev genus one classification result. Then the determinant of L is

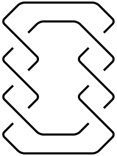
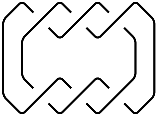
$$\left|\sum_{i=1}^{2k}(-1)^i \det D(R_1)\cdots\det D(R_{i-1})\det N(R_i)\det D(R_{i+1})\cdots\det D(R_{2k})\right|.$$

- Generalizes formula for the determinant of Montesinos links.
- Related to the formula for the determinant of Turaev genus one links by DFKLS.

A determinant example

det 

$=$ det  $\cdot$ det 

$-$ det  $\cdot$ det 

$$= 9 \cdot 9 - 6 \cdot 6 = 45$$

A signature formula

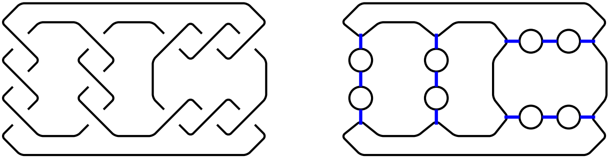
Theorem (Dasbach, L.)

Let K be a knot with diagram D as in the Turaev genus one classification result. Let $s_A(D)$ be the number of components in the all-A resolution of D, and let $c_+(D)$ be the number of positive crossings in D. The signature of K is determined by

$$\sigma(K) = s_A(D) - c_+(D) \pm 1 \text{ and } \sigma(K) \equiv \det(K) - 1 \mod 4.$$

- If K has alternating diagram D, then Traczyk showed that $\sigma(K) = s_A(D) - c_+(D) - 1$.
- The above result follows from some earlier work relating Turaev genus to signature.

A signature example



$$\sigma(K) = s_A(D) - c_+(D) \pm 1$$
$$= 9 - 0 \pm 1$$
$$= 8 \text{ or } 10.$$
$$\det(K) = 45.$$
$$8 \equiv 45 - 1 \mod 4.$$
$$\sigma(K) = 8.$$

Signature: Part two

Theorem (Dasbach, L.)

Let L be a link with Turaev genus one diagram D as in the classification result. Either

$$\sigma(L) = \pm 1 + \sum_{i=1}^{2k} \sigma(N(R_i)) \text{ or}$$

$$\sigma(L) = \pm 1 + \sum_{i=1}^{2k} \sigma(D(R_i)),$$

where the orientation of L determines the choice.

Almost-alternating links

- A link diagram D is almost-alternating if one crossing change transforms D into an alternating diagram (Adams, Brock, Bugbee - 1992).
- A link L is almost-alternating if it is non-alternating and has an almost-alternating diagram.
- If L is almost-alternating, then $g_T(L) = 1$.

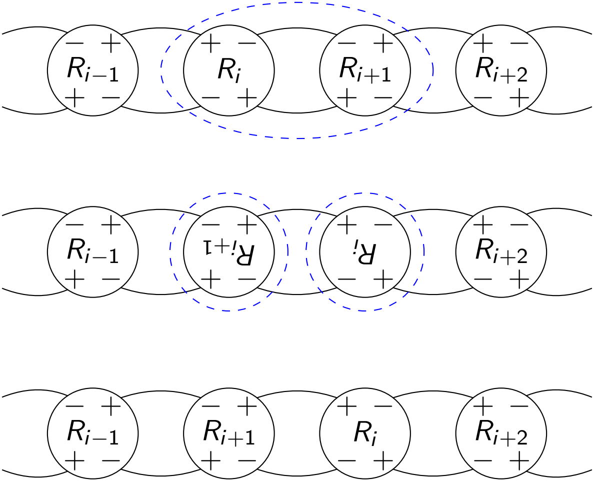
Mutation

- Let B be a 3-ball whose boundary intersects the link L in exactly 4-points. A mutation of L is a link obtained by removing B from S^3, rotating it 180° about a principle axis, and then gluing B back into S^3.
- Any two links related to one another via a sequence of mutations are said to be mutant to one another.

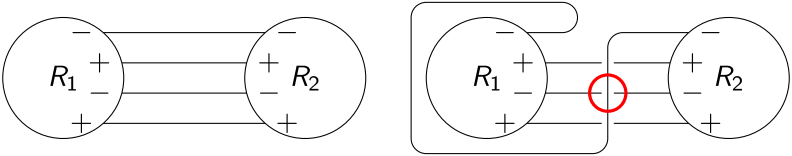
Theorem (Armond, L.)

If $g_T(L) = 1$, then L is mutant to an almost-alternating link.

Mutation proof



Mutation proof continued



Jones polynomial results

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

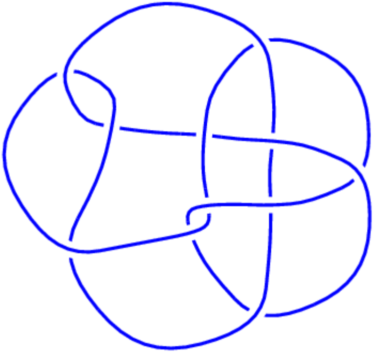
Theorem (Dasbach, L.)

If L is almost-alternating, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

Corollary

If $g_T(L) = 1$, then either $|a_m| = 1$ or $|a_n| = 1$ (or both equal 1).

An example: $11n_{95}$



$$V_{11n_{95}}(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9$$

Low crossing results

- Among all non-alternating knots with 12 or fewer crossings, it is unknown whether 37 of them have Turaev genus one (according to Jablan's and Howie's computations on Knot Info).
- Our work shows that 12 of these 37 knots do not have Turaev genus one (and thus have Turaev genus two).

Thank you!