Invariants of Turaev genus one links

Adam Lowrance - Vassar College
Oliver Dasbach - Louisiana State University

March 9, 2017

Philosophy

1. Start with a family of links $\mathcal{F}$ and a link invariant $\mathrm{Inv}(L)$.
2. Prove that if $L \in \mathcal{F}$, then $\mathrm{Inv}(L)$ is special in some way.
3. Use (2) to create a membership test for $\mathcal{F}$.
4. Use (2) to determine properties of links in $\mathcal{F}$.

Philosophy in action

Let $\mathcal{F}$ be the class of alternating links, and let $\mathrm{Inv}(L)$ be the Jones polynomial $V_L(t)$.

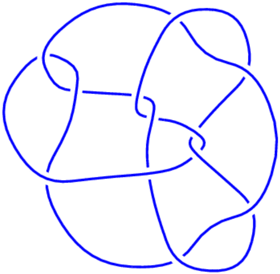
Theorem (Kauffman, Murasugi, Thistlethwaite - 1987)

Let L be an alternating link. The Jones polynomial of L has the form

$$V_L(t) = \sum_{i=m}^{n} a_i t^i$$

where $m, n \in \frac{1}{2}\mathbb{Z}$, $n - m = c(L)$ the minimum crossing number of L, and $|a_m| = |a_n| = 1$.

Membership test: Is $12n_{375}$ alternating?



Since

$$V_{12n_{375}}(t) = 2t^2 - 4t^3 + 8t^4 - 9t^5 + 10t^6 - 10t^7 + 7t^8 - 5t^9 + 2t^{10},$$

the knot $12n_{375}$ above is not alternating.

Properties of alternating knots determined by $V_L(t)$

The Kauffman, Murasugi, and Thistlethwaite theorem implies that the span of the Jones polynomial equals the minimal crossing number of an alternating link.

It also implies that the only alternating knot whose Jones polynomial is 1 is the unknot.

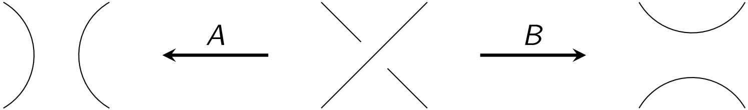
Our setting today

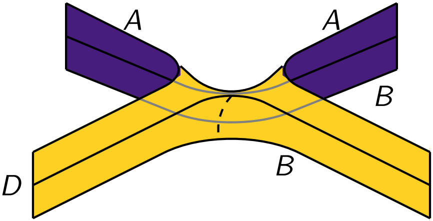
- The invariants we consider are the Jones polynomial $V_L(t)$ of L and the Khovanov homology $Kh(L)$ of L.
- The families of links that we consider are links of Turaev genus one and almost alternating links.
- The Jones polynomial and Khovanov homology give membership tests to these two families.

Construction of the Turaev surface of D

1. Replace arcs of D not near crossings with bands transverse to the projection plane.
2. Replace crossings of D with saddles interpolating between the all-B and all-A states of D
3. Cap off the boundary components with disks to obtain the Turaev surface of D.

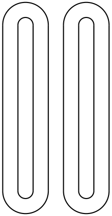
The Turaev surface in pictures

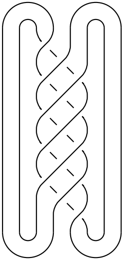




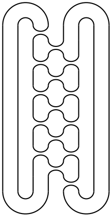
The Turaev surface of the (4, 5)-torus knot

all-A state

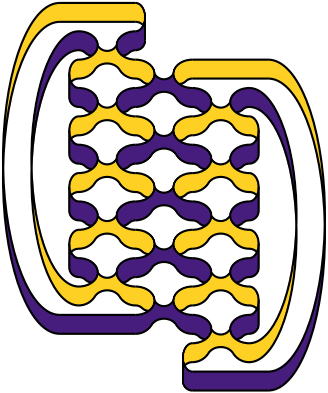




D



all-B state



Turaev surface of D

Turaev genus

- For a diagram D of a link L, let $g_T(D)$ denote the genus of the Turaev surface.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

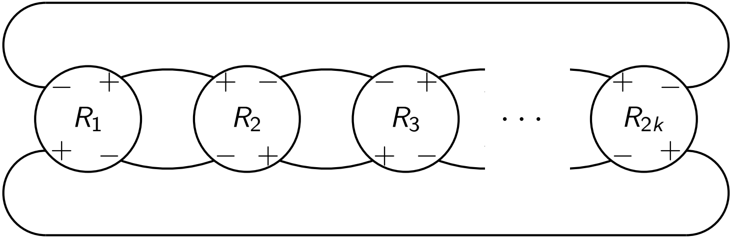
Some history of the Turaev surface

- Turaev (1987) related the Turaev surface to the difference between span $V_L(t)$ and the crossing number.
- DFKLS (2006) - The Jones polynomial is an evaluation of the Bollobás-Riordan-Tutte polynomial of a graph embedded on the Turaev surface.
- CKS (2007), DL (2007, 2009, 2015) - Connections with Khovanov homology and knot Floer homology.
- Kalfagianni (2016) - Characterization of adequate knots using the colored Jones polynomial and Turaev genus.

Computations of Turaev genus

- Turaev (1987) - A link is alternating if and only if $g_T(L) = 0$.
- Non-alternating pretzel and Montesinos links are of Turaev genus one.
- Abe, Kishimoto (2008) - Computed the Turaev genus of three-stranded torus knots.
- Abe (2009) - An adequate diagram is Turaev genus minimizing.
- Jin, L., Polston, Zheng (2017) - Computed the Turaev genus of many torus knots with six or fewer strands.

Classification of links of Turaev genus one



Theorem (Armond, L.; Kim - 2015)

1. *Let $R_1, \ldots, R_{2k}$ be alternating two-tangles, and let D be a link diagram connecting $R_1, \ldots, R_{2k}$ as depicted above. Then $g_T(D) = 1$.*
2. *Moreover, if L is a non-split link with $g_T(L) = 1$, then L has a diagram as above.*

Almost alternating links

- A link diagram D is almost alternating if one crossing change transforms D into an alternating diagram (Adams, Brock, Bugbee, Comar, Faigin, Huston, Joseph, Pesikoff - 1992).
- A link L is almost alternating if it is not alternating and has an almost alternating diagram.
- If L is almost alternating, then $g_T(L) = 1$. The converse is open.

Mutation

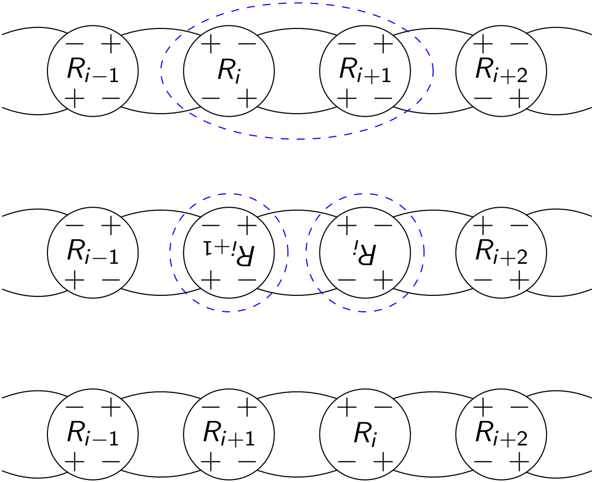
- Let B be a 3-ball whose boundary intersects the link L in exactly 4-points. A mutation of L is a link obtained by removing B from S^3, rotating it 180° about a principle axis, and then gluing B back into S^3.

- Any two links related to one another via a sequence of mutations are said to be mutant to one another.

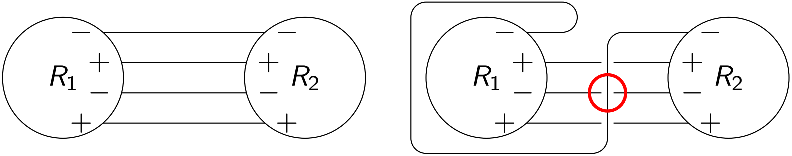
Theorem (Armond, L. - 2015)

If $g_T(L) = 1$, then L is mutant to an almost alternating link.

Mutation proof



Mutation proof continued



Jones polynomial results

Let $V_L(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots + a_{n-1} t^{n-1} + a_n t^n$ be the Jones polynomial of L where a_m and a_n are nonzero.

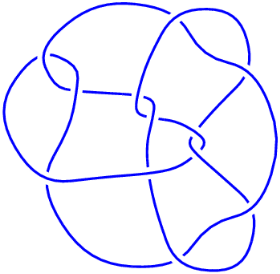
Theorem (Dasbach, L. - 2016)

If L is almost-alternating, then $|a_m| = 1$ or $|a_n| = 1$.

Corollary

If $g_T(L) = 1$, then $|a_m| = 1$ or $|a_n| = 1$.

$12n375$ returns



Since

$$V_{12n_{375}}(t) = 2t^2 - 4t^3 + 8t^4 - 9t^5 + 10t^6 - 10t^7 + 7t^8 - 5t^9 + 2t^{10},$$

the knot $12n_{375}$ is neither almost alternating nor Turaev genus one.

Khovanov homology

Let $j_{\min}$ and $j_{\max}$ be the least and greatest polynomial grading where the Khovanov homology of L is non-trivial.

Theorem (Dasbach, L.)

If L is almost alternating or Turaev genus one, then either $Kh^{*,j_{\min}}(L)$ *or* $Kh^{*,j_{\max}}(L)$ *is isomorphic to* $\mathbb{Z}$.

Crossing number of almost alternating links

Theorem (Adams et al. - 1992)

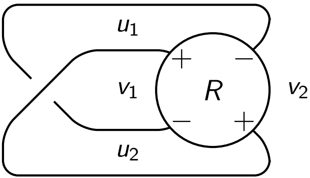
Let D be an almost alternating diagram of a link L with c crossings. Then

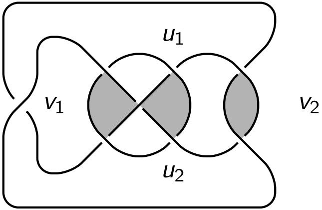
$$\operatorname{span} V_L(t) \leq c - 3.$$

If $\operatorname{span} V_L(t) = c - 3$, *then D has the fewest number of crossings among all almost alternating diagrams of L.*

Adjacent faces

A face f in R is adjacent to faces u_1 and u_2 if it is in the same checkerboard graph as u_1 and u_2 and shares crossings with both. Likewise define faces adjacent to v_1 and v_2. Let $\mathrm{adj}(u_1, u_2)$ and $\mathrm{adj}(v_1, v_2)$ be the number of adjacent faces of each type.





$$\mathrm{adj}(u_1, u_2) = 3 \text{ and } \mathrm{adj}(v_1, v_2) = 0.$$

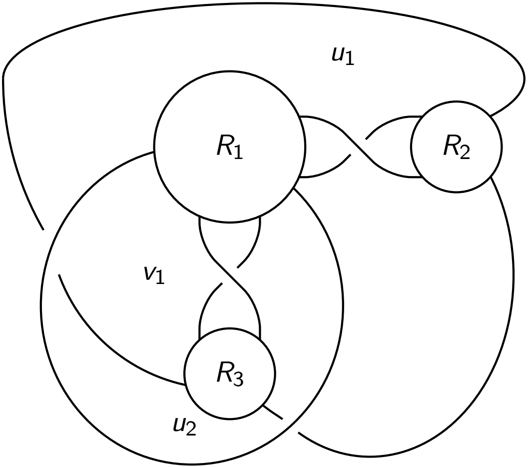
Minimal crossing almost alternating diagrams

Let D be an almost alternating diagram of a link L with c crossings.

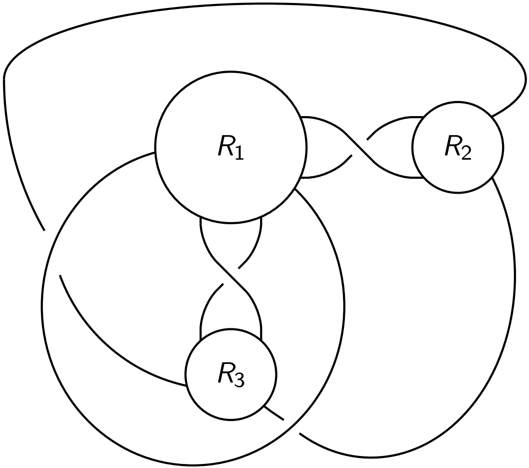
1. If $\text{adj}(u_1, u_2) \neq 1$ and $\text{adj}(v_1, v_2) \neq 1$, then D has the fewest number of crossings among all almost alternating diagrams of L.
2. If $\text{adj}(u_1, u_2) = 1$ and $\text{adj}(v_1, v_2) = 1$, then D has an almost alternating diagram with $c - 2$ crossings.
3. Suppose exactly one of $\text{adj}(u_1, u_2)$ and $\text{adj}(v_1, v_2)$ is one. This case is in progress with Spyropoulos.

Proof of item 2

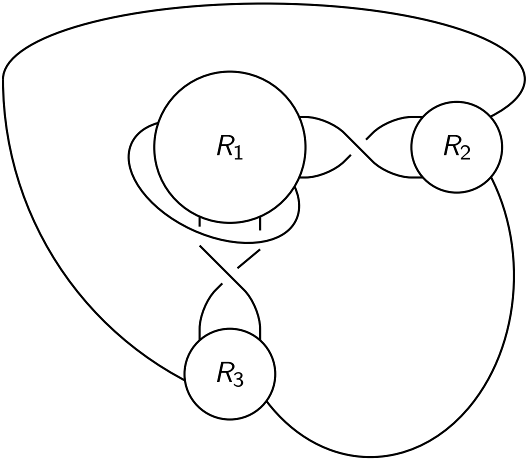
If $\mathrm{adj}(u_1, u_2) = \mathrm{adj}(v_1, v_2) = 1$, then D has diagram as below.



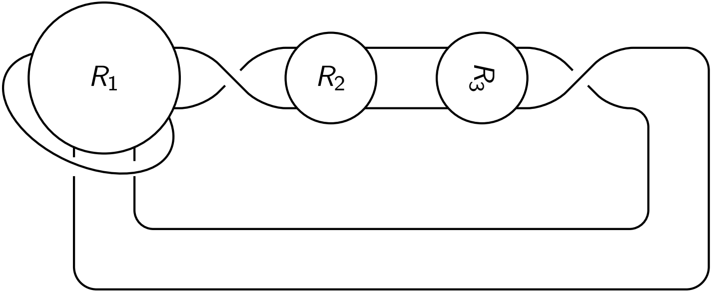
Proof of item 2



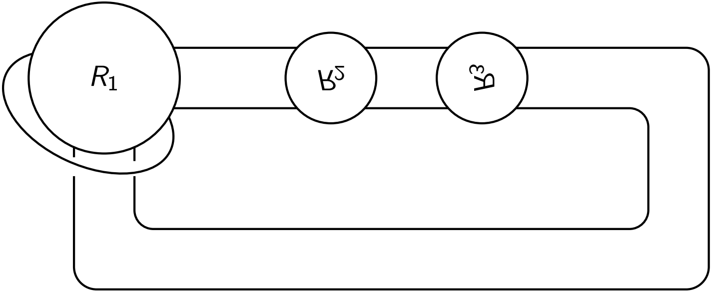
Proof of item 2



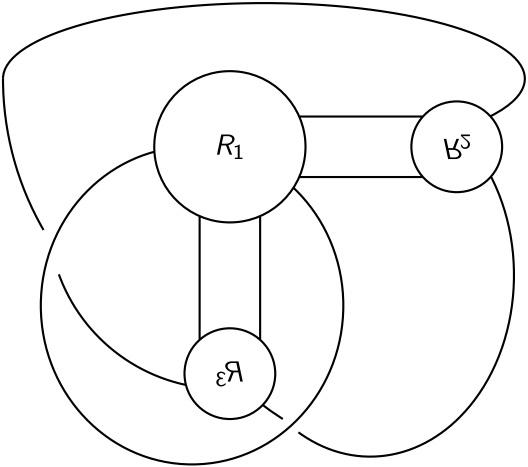
Proof of item 2



Proof of item 2



Proof of item 2



Current and future directions

1. Is there a nontrivial almost alternating knot whose Jones polynomial is 1?
2. Can the coefficients of the Jones polynomial of an almost alternating or Turaev genus one knot give information about the hyperbolic volume of the knot?
3. Can we say more about the Khovanov homology of almost alternating links?

Thank you!