Alternating distances of knots and links

Adam Lowrance - Vassar College

July 24, 2014
Alternating distance

An integer valued knot invariant \(d(K)\) is an *alternating distance* if

- \(d(K) \geq 0\) for every knot \(K\),
- \(d(K) = 0\) if and only if \(K\) is alternating, and
- \(d(K_1 \# K_2) \leq d(K_1) + d(K_2)\) for all knots \(K_1\) and \(K_2\).
Let $d_1$ and $d_2$ be alternating distances, and let $\mathcal{F}$ be some family of knots. We say $d_1$ dominates $d_2$ on $\mathcal{F}$ if for each positive integer $n$, there exists a knot $K_n \in \mathcal{F}$ such that

$$d_1(K_n) - d_2(K_n) \geq n.$$
The Turaev surface

- Knot diagram $D \leadsto$ Turaev surface $F(D)$.
- Turaev (1987) used $F(D)$ to give simplified proofs of some Tait conjectures.
- DFKLS (2006) - Connections with the Jones polynomial.
- Dasbach, L. (2009) - Connections with signature, and the $s$ and $\tau$ invariants.
Construction of the Turaev surface $F(D)$

1. Replace crossings of $D$ with disks.
2. Replace strands of $D$ between crossings with (sometimes twisted) bands.
3. Cap off the boundary components with disks to obtain $F(D)$. 

The Turaev surface - in pictures
An alternating example
An alternating example
An alternating example
An alternating example
A non-alternating example
A non-alternating example
A non-alternating example
A non-alternating example
The Turaev genus $g_T(K)$ of the knot $K$ is

$$g_T(K) = \min\{g(F(D)) \mid D \text{ is a diagram of } K\}.$$ 

**Theorem (DFKLS)**

The Turaev genus of $K$ is zero if and only if $K$ is alternating. Moreover, the Turaev genus of a knot is an alternating distance.
More facts about the Turaev surface

1. $F(D)$ is a Heegaard surface in $S^3$.

2. $D$ is alternating on $F(D)$.

3. The complement of $D$ in $F(D)$ is a collection of disks that can be two-colored in a checkerboard fashion.

**Question.** If a surface satisfies (1) - (3) above, is it the Turaev surface of some knot?

**Answer.** No.
Alternating genus of a knot

- Let $g_{alt}(K)$ denote the minimum genus surface satisfying (1)-(3) from the previous slide.
- Knots $K$ with $g_{alt}(K) = 1$ were studied by Adams (1994).
- The alternating genus of a knot is an alternating distance.
- $g_{alt}(K) \leq g_T(K)$ for any knot, but there exists knots where $g_T(K)$ is much larger than $g_{alt}(K)$. 
A modified torus knot $\tilde{T}_{4,4k+3}$

Let $B_4$ be the braid group on 4-strands, and let $\Delta \in B_4$ denote the braid

$$\Delta = \sigma_1 \sigma_2 \sigma_3.$$ 

Let $\tilde{\Delta}$ denote the braid

$$\tilde{\Delta} = \sigma_1 \sigma_2^{-1} \sigma_3.$$ 

For each non-negative integer $k$, define $\tilde{T}_{4,4k+3}$ to be the closure of the braid

$$\Delta^{4k+2} \tilde{\Delta}.$$ 

Let $\mathcal{F}(\tilde{T}_{4,4k+3})$ be the set of such modified torus knots.
Example: $\tilde{T}_{4,3}$
Example: $\tilde{T}_{4,3}$
Turaev genus dominates alternating genus

Theorem

*Turaev genus dominates alternating genus on* $F(\tilde{T}_{4,4k+3})$.

Overview of proof.

• $g_{alt}(\tilde{T}_{4,4k+3}) = 1$ for any non-negative integer $k$.

• $g_T(\tilde{T}_{4,4k+3}) \to \infty$ as $k \to \infty$. 
The alternating genus of $\tilde{T}_{4,4k+3}$
The alternating genus of $\tilde{T}_{4,4k+3}$
The alternating genus of $\tilde{T}_{4,4k+3}$
The alternating genus of $\tilde{T}_{4,4k+3}$
The Turaev genus of $\tilde{T}_{4,4k+3}$

1. Dasbach, L. (2009) For any knot $K$,
   \[ |s(K) + \sigma(K)| \leq 2g_T(K). \]

2. Use Gordon, Litherland, and Murasugi (1981) to compute $\sigma(\tilde{T}_{4,4k+3})$.

3. The Rasmussen $s$ invariant and the signature of a knot can change by at most two for each crossing change.

4. Use (2) and (3) to show that $|s(\tilde{T}_{4,4k+3}) + \sigma(\tilde{T}_{4,4k+3})| \to \infty$ as $k \to \infty$.

5. Thus (1) implies $g_T(\tilde{T}_{4,4k+3}) \to \infty$ as $k \to \infty$. 
Recall that the Turaev surface $F(D)$ satisfies the following.

1. $F(D)$ is a Heegaard surface in $S^3$.
2. $D$ is alternating on $F(D)$.
3. The complement of $D$ in $F(D)$ is a collection of disks that can be two-colored in a checkerboard fashion.

Sufficient conditions for being a Turaev surface.

- Champanerkar, Kofman (2014) - Conditions (1)-(3) plus a Morse theoretic condition.
- Armond, Druivenga, Kindred (in progress) - Conditions (1)-(3) plus a Heegaard diagram condition.
The dealternating number

Let $D$ be a diagram of $K$. The dealternating number of $D$, denoted $\text{dalt}(D)$, is the minimum number of crossing changes required to transform $D$ into an alternating diagram.

$\text{dalt}(D) = 1$

Adams (1992) defines the dealternating number of $K$ as

$$\text{dalt}(K) = \min\{\text{dalt}(D) \mid D \text{ is a diagram of } K\}.$$
Let $D$ be a diagram of $K$. The alternation number of $D$, denoted $\text{alt}(D)$, is the minimum number of crossing changes required to transform $D$ into a (possibly non-alternating) diagram of an alternating knot.

\[ \text{alt}(D) = 0 \]

Kawauchi (2010) defines the alternation number of $K$ as

\[ \text{alt}(K) = \min\{\text{alt}(D) \mid D \text{ is a diagram of } K\}. \]
More alternating distances

- The dealternating number of a knot is an alternating distance.
- The alternation number of a knot is an alternating distance.
- Let $c(K)$ denote the crossing number of $K$, and let $V_K(t)$ denote the Jones polynomial of $K$. Then

$$c(K) - \text{span } V_K(t)$$

is an alternating distance (Kauffman - 1987, Murasugi - 1987).
Inequalities

The following inequalities hold for every knot $K$:

1. $g_{alt}(K) \leq g_T(K)$.

2. $alt(K) \leq dalt(K)$.

3. $g_T(K) \leq dalt(K)$ (Abe - 2008).

4. $g_T(K) \leq c(K) - \text{span } V_K(t)$ (Turaev - 1987).
Our inequalities

dalt(K) \quad c(K) - \text{span } V_K(t)

alt(K) \quad g_T(K) \quad g_{alt}(K)
The $t$-twisted positive Whitehead double of $K$
Let $W_n$ be the $n$-th iterated, positive, untwisted Whitehead double of the figure-eight knot. Let $F(W_n)$ be the set of all $W_n$.

**Theorem**

*Turaev genus dominates alternation number on $F(W_n)$.*

**Proof sketch.**

- $\text{alt}(W_n) = 1$ for each $n > 0$.
- Knot Floer homology tells us $g_T(W_n) \to \infty$ as $n \to \infty$. 
Alternation number and alternating genus

Theorem
*The alternation number and alternating genus of a knot are not comparable.*

Proof sketch.
- Using a similar argument from before, we have
  \[ g_{\text{alt}}(\tilde{T}_{4,4k+3}) < \text{alt}(\tilde{T}_{4,4k+3}). \]
- Adams (1994) proves that \( g_{\text{alt}}(W_n) > 1 \) and so
  \[ \text{alt}(W_n) < g_{\text{alt}}(W_n). \]
The difference $c(K) - \text{span } V_K(t)$ dominates the dealternating number on the set of $(3, q)$-torus knots.

Open Questions

1. Are the Turaev genus and the dealternating number the same invariant, i.e. does \( g_T(K) = \text{dalt}(K) \) for every knot \( K \)?

2. Is \( \text{dalt}(K) \leq c(K) - \text{span} \ V_K(t) \) for every knot \( K \)?

3. Is \( \text{alt}(K) \leq c(K) - \text{span} \ V_K(t) \) for every knot \( K \)?

4. Is \( \text{alt}(K) \leq g_T(K) \) for every knot \( K \)?