Alternating distances of knots

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Alternating distance

An integer valued knot invariant $d(K)$ is an *alternating distance* if

- $d(K) \geq 0$ for every knot K,
- $d(K) = 0$ if and only if K is alternating, and
- $d(K_1 \# K_2) \leq d(K_1) + d(K_2)$ for all knots K_1 and K_2.

Three alternating distances

- The Turaev genus $g_T(K)$ of the knot K.
- The alternating genus $g_{\text{alt}}(K)$ of the knot K.
- The dealternating number $\text{dalt}(K)$ of the knot K.

The Turaev surface

- Knot diagram $D \rightsquigarrow$ Turaev surface $F(D)$.
- Turaev (1987) used $F(D)$ to give simplified proofs of some Tait conjectures.
- DFKLS (2006) - Connections with the Jones polynomial.
- Champanerkar, Kofman, and Stoltzfus (2007) - Connections with Khovanov homology.
- L. (2007) - Connections with knot Floer homology.
- Dasbach, L. (2009) - Connections with signature, and the s and τ invariants.

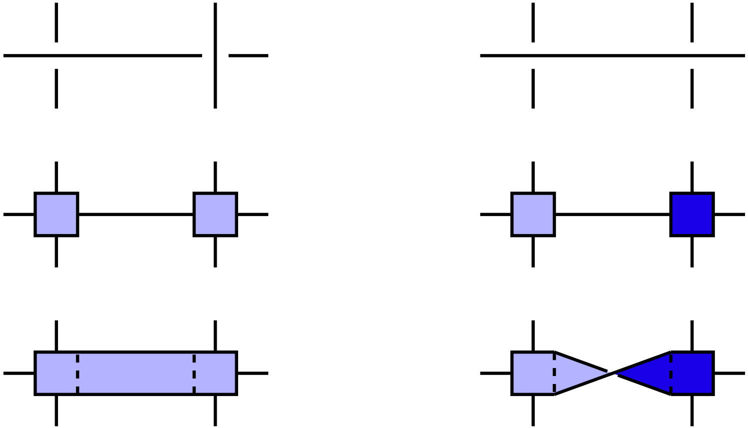
Construction of the Turaev surface $F(D)$

1. Replace crossings of D with disks.

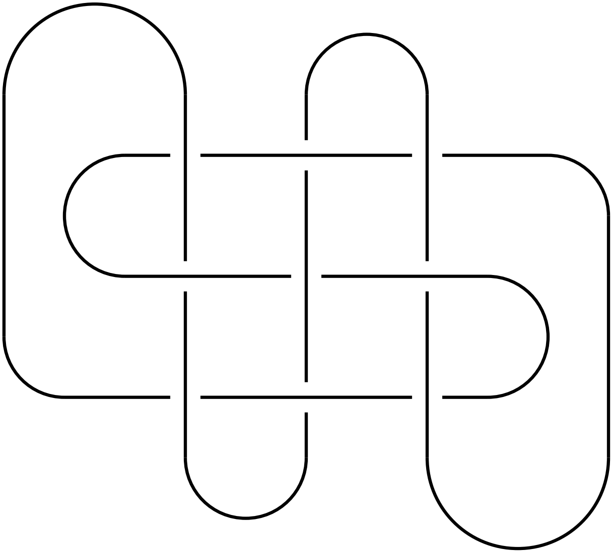
2. Replace strands of D between crossings with (sometimes twisted) bands.

3. Cap off the boundary components with disks to obtain $F(D)$.

The Turaev surface - in pictures



An alternating example



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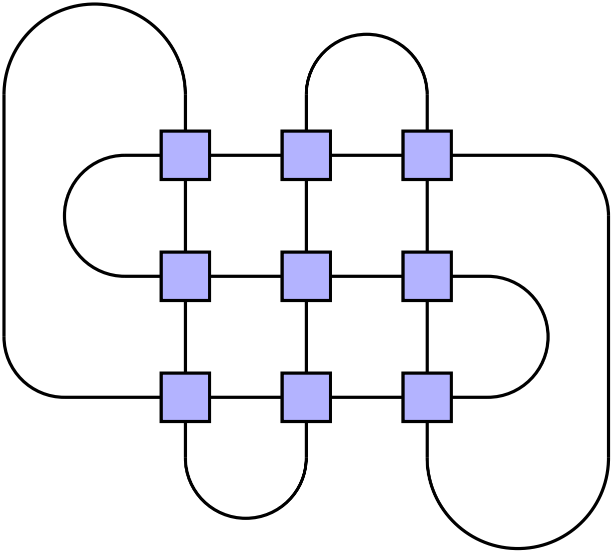

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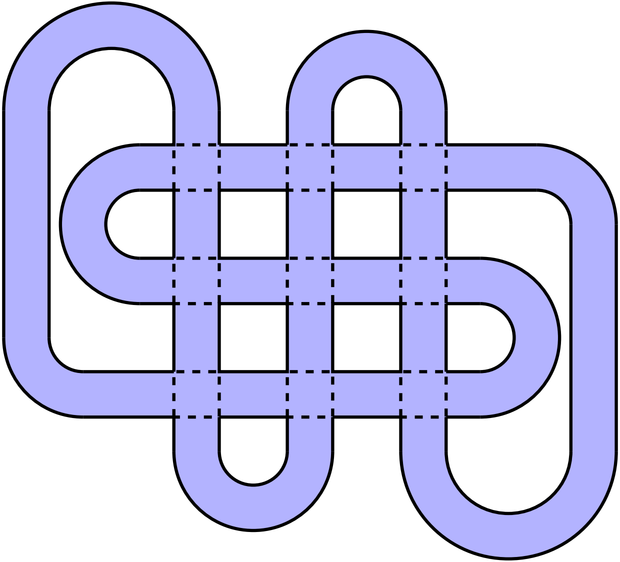

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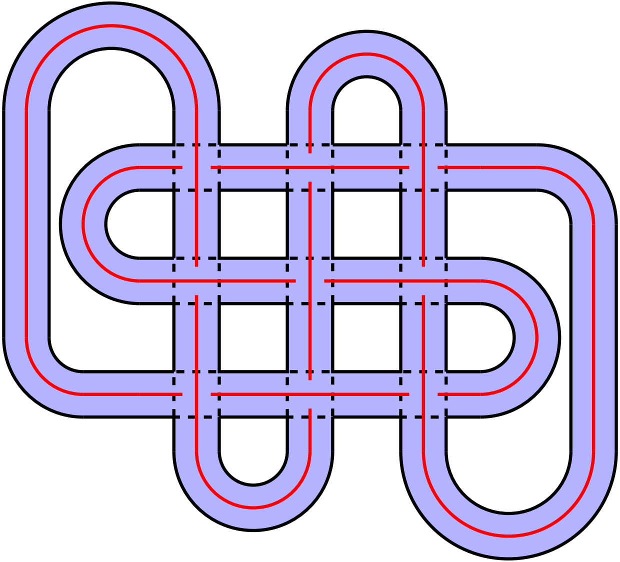
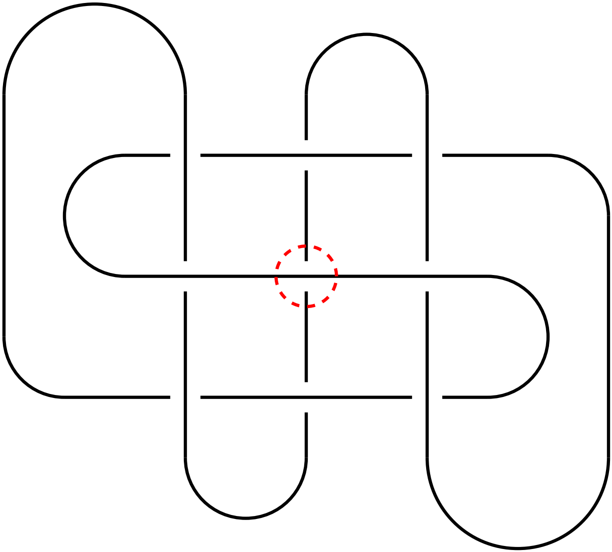


A non-alternating example



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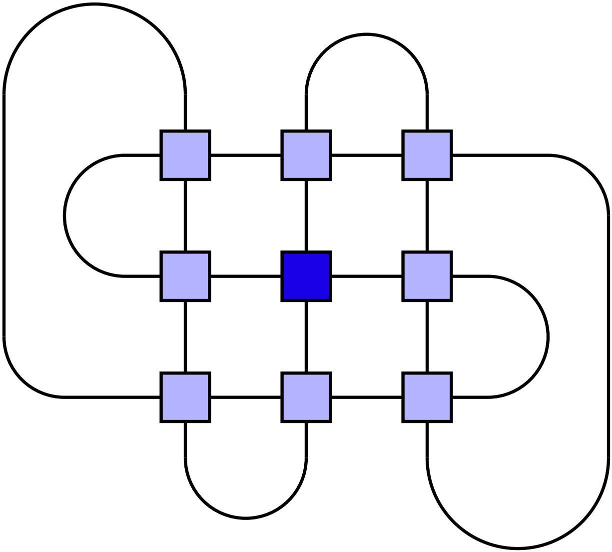

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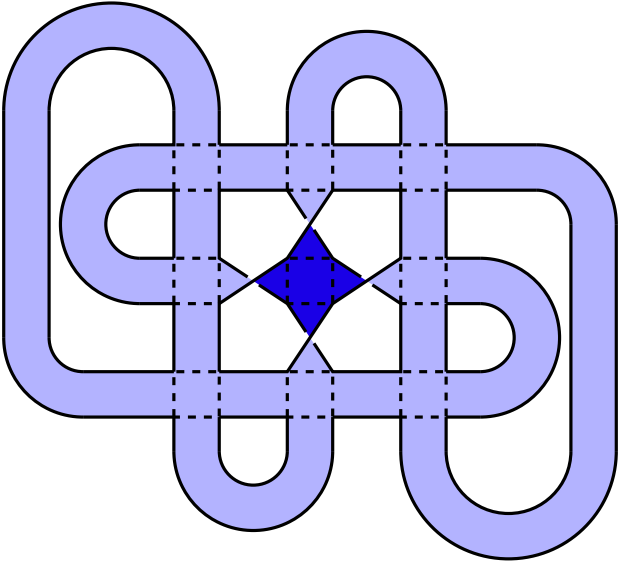

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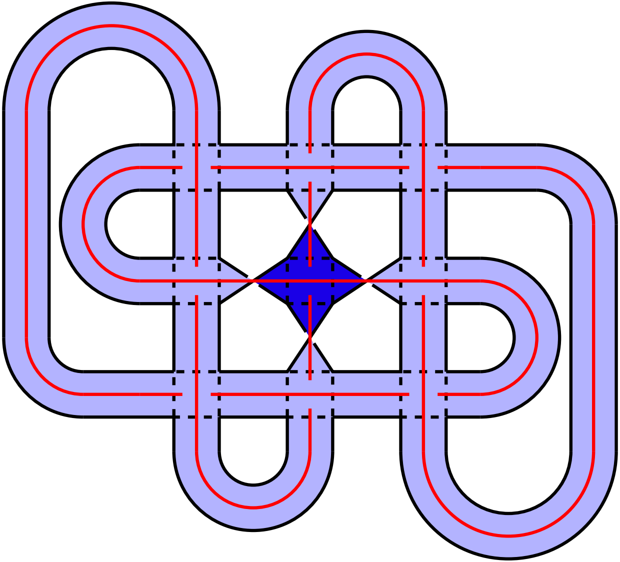


Turaev genus

- For D a diagram of a knot K, let $g_T(D)$ be the genus of the Turaev surface $F(D)$.

- The Turaev genus $g_T(K)$ of the knot K is

$$g_T(K) = \min\{g_T(D) \mid D \text{ is a diagram of } K\}.$$

Turaev genus is an alternating distance

Theorem (Turaev)

The Turaev genus is an alternating distance, i.e.

- $g_T(K) = 0$ *if and only if* K *is alternating,*
- $g_T(K) \geq 0$ *for all knots* K*, and*
- $g_T(K_1 \# K_2) \leq g_T(K_1) + g_T(K_2)$ *for all knots* K_1 *and* K_2*.*

Properties of the Turaev surface

1. $F(D)$ is a Heegaard surface in S^3.
2. D is alternating on $F(D)$.
3. The complement of D in $F(D)$ is a collection of disks that can be two-colored in a checkerboard fashion.

Question. If a surface satisfies (1) - (3) above, is it the Turaev surface of some knot?

The question rephrased

Let K be a knot and let F be a surface such that

1. F is a Heegaard surface of S^3,
2. K has an alternating projection to F, and
3. the complement of the projection of K is a collection of disks.

Question. If K and F satisfy the above conditions, then is F the Turaev surface of some diagram D of K?
Answer. No.

Define the alternating genus $g_{\text{alt}}(K)$ of K to be the minimum genus surface satisfying (1)-(3) above.

Alternating genus of a knot

- Knots K with $g_{\text{alt}}(K) = 1$ were studied by Adams (1994).
- The additivity of alternating genus under connect sum was studied by Balm (2013).
- The alternating genus of a knot is an alternating distance.
- $g_{\text{alt}}(K) \leq g_T(K)$ for any knot, but there exists knots where $g_T(K)$ is much larger than $g_{\text{alt}}(K)$.

A modified torus knot $\widetilde{T}_{4,4k+3}$

Let B_4 be the braid group on 4-strands, and let $\Delta \in B_4$ denote the braid

$$\Delta = \sigma_1 \sigma_2 \sigma_3.$$

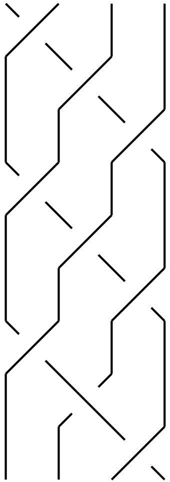
Let $\widetilde{\Delta}$ denote the braid

$$\widetilde{\Delta} = \sigma_1 \sigma_2^{-1} \sigma_3.$$

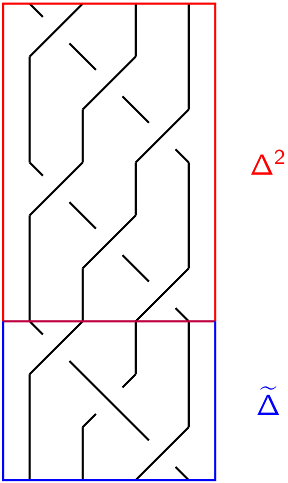
For each non-negative integer k, define $\widetilde{T}_{4,4k+3}$ to be the closure of the braid

$$\Delta^{4k+2} \widetilde{\Delta}.$$

Example: $\widetilde{T}_{4,3}$



Example: $\widetilde{T}_{4,3}$



Turaev genus vs. alternating genus

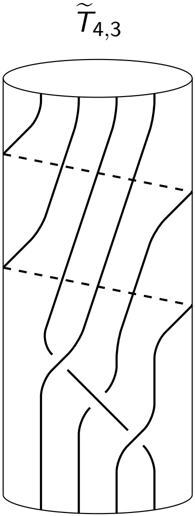
Theorem

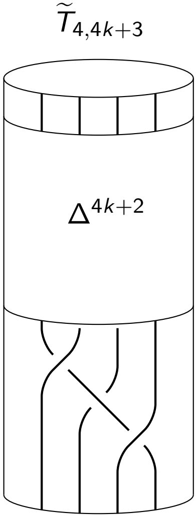
The difference $g_T(\widetilde{T}_{4,4k+3}) - g_{alt}(\widetilde{T}_{4,4k+3}) \to \infty$ *as* $k \to \infty$.

Overview of proof.

- $g_{\text{alt}}(\widetilde{T}_{4,4k+3}) = 1$ for any non-negative integer k.
- $g_T(\widetilde{T}_{4,4k+3}) \to \infty$ as $k \to \infty$.

The alternating genus of $\widetilde{T}_{4,4k+3}$





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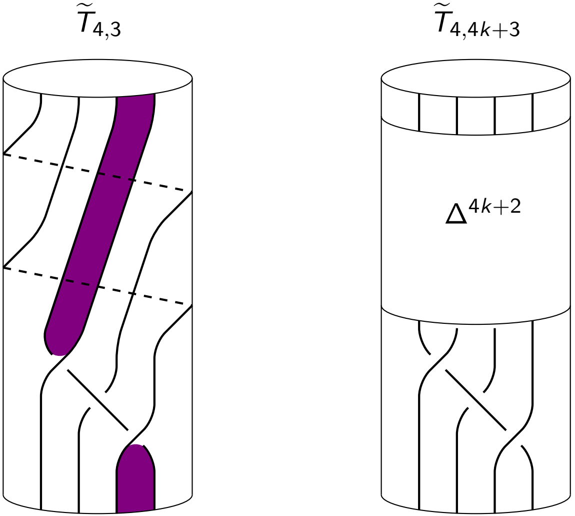

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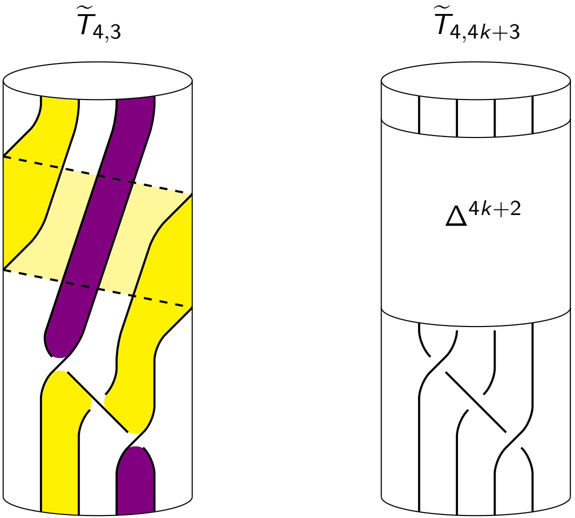
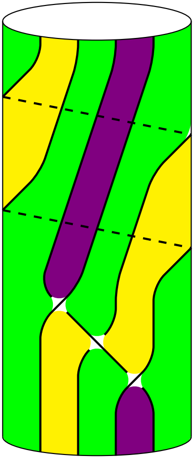
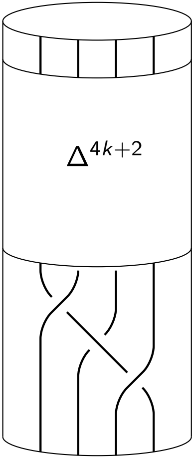


The alternating genus of $\widetilde{T}_{4,4k+3}$

$\widetilde{T}_{4,3}$



$\widetilde{T}_{4,4k+3}$



The Turaev genus of $\widetilde{T}_{4,4k+3}$

1. Dasbach, L. (2009) For any knot K,
$$|s(K) + \sigma(K)| \leq 2g_T(K).$$
2. Use Gordon, Litherland, and Murasugi (1981) to compute $\sigma(T_{4,4k+3})$.
3. The Rasmussen s invariant and the signature of a knot can change by at most two for each crossing change.
4. Use (2) and (3) to show that $|s(\widetilde{T}_{4,4k+3}) + \sigma(\widetilde{T}_{4,4k+3})| \to \infty$ as $k \to \infty$.
5. Thus (1) implies $g_T(\widetilde{T}_{4,4k+3}) \to \infty$ as $k \to \infty$.

Sufficient conditions for being a Turaev surface

Suppose K is a knot and F is a surface satisfying

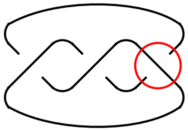
1. F is a Heegaard surface of S^3,
2. K has an alternating projection to F, and
3. the complement of the projection of K is a collection of disks.

Two groups give additional conditions that ensure F is a Turaev surface:

- Champanerkar, Kofman (2014) - Conditions (1)-(3) plus a Morse theoretic condition.
- Armond, Druivenga, Kindred (2014) - Conditions (1)-(3) plus a Heegaard diagram condition.

The dealternating number

Let D be a diagram of K. The dealternating number of D, denoted $\text{dalt}(D)$, is the minimum number of crossing changes required to transform D into an alternating diagram.



$\text{dalt}(D) = 1$

Adams (1992) defines the dealternating number of K as

$$\text{dalt}(K) = \min\{\text{dalt}(D) \mid D \text{ is a diagram of } K\}.$$

Turaev genus vs. dealternating number

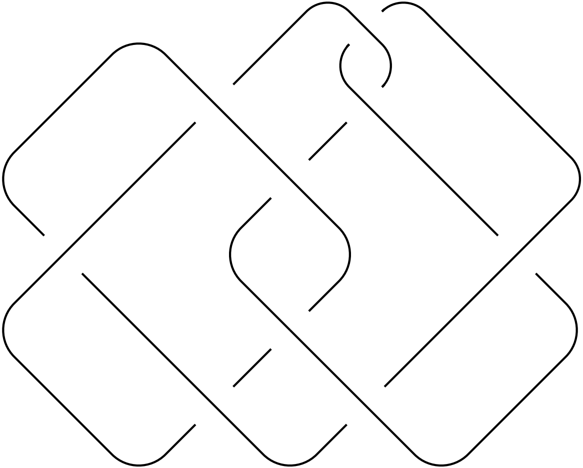
Theorem (Abe, Kishimoto)

Let K be a knot. Then $g_T(K) \leq dalt(K)$.

Sketch of proof. The genus of the Turaev surface $g_T(D)$ changes by at most one for each crossing change. Hence $g_T(D) \leq \text{dalt}(D)$ for every diagram D.

Question. Is there a knot K with $g_T(K) < \text{dalt}(K)$?

An example: 9_{42}



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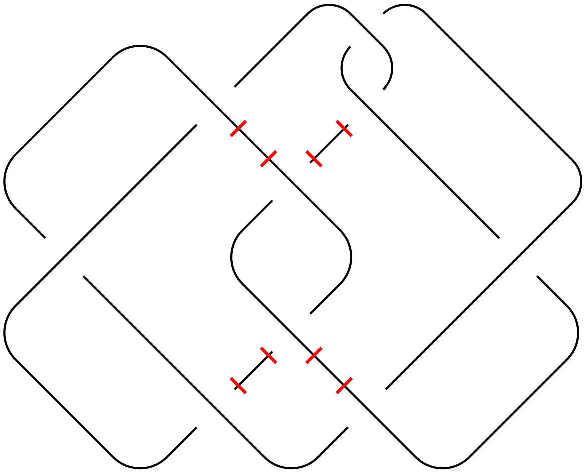

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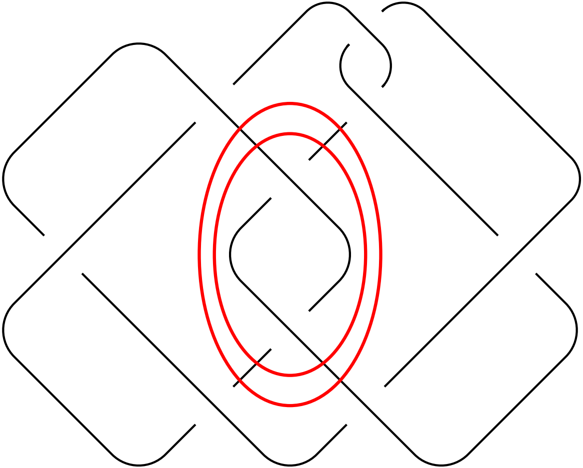
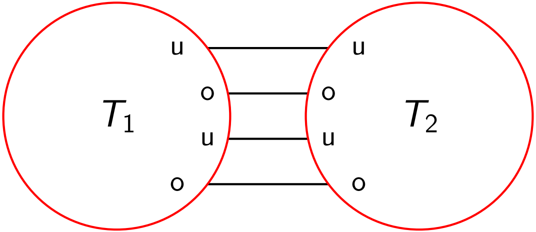


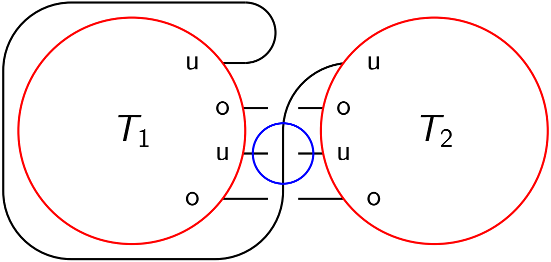
Alternating tangle decomposition



Glue together alternating tangles T_1 and T_2 to get a diagram D. Then

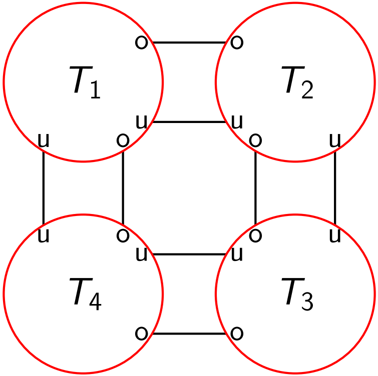
- $g_T(D) = 1$, but
- $\mathrm{dalt}(D) = \min\{c(T_1), c(T_2)\}$.

Modified alternating tangle decomposition



A different diagram D' of the same knot has $\text{dalt}(D') = 1$.

Longer strings of alternating tangles



The Turaev genus of any such string of $2k$ alternating tangles is one. Is the dealternating number of any such knot also one?

Thank you!