Turaev genus and alternating decompositions

Adam Lowrance - Vassar College
Joint with Cody Armond - University of Iowa

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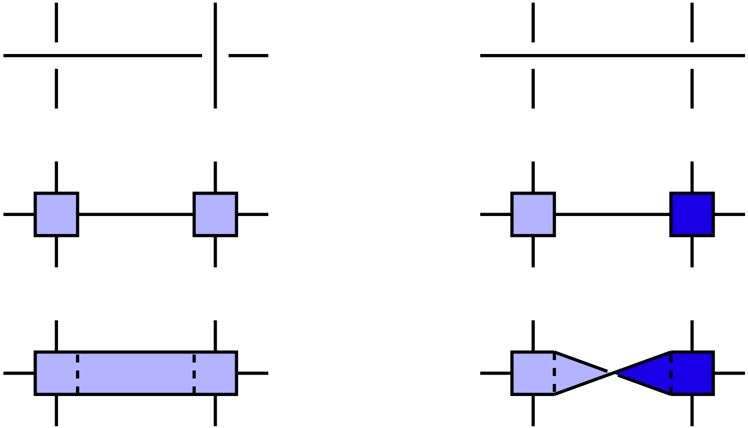
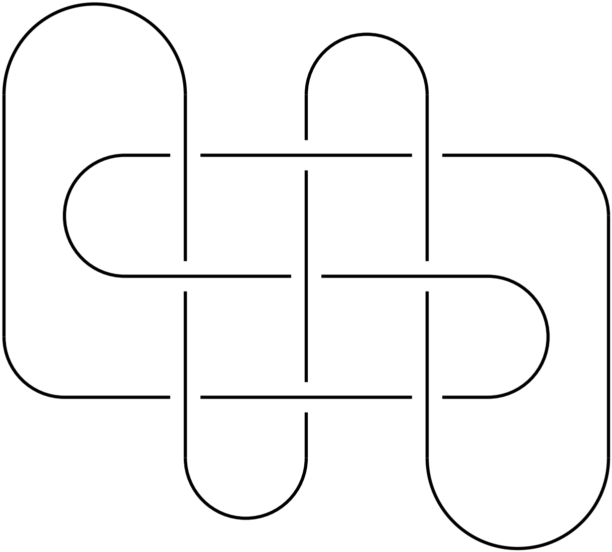
Construction of the Turaev surface $F(D)$

1. Replace crossings of D with disks.
2. Replace strands of D between crossings with (sometimes twisted) bands.
3. Cap off the boundary components with disks to obtain $F(D)$.

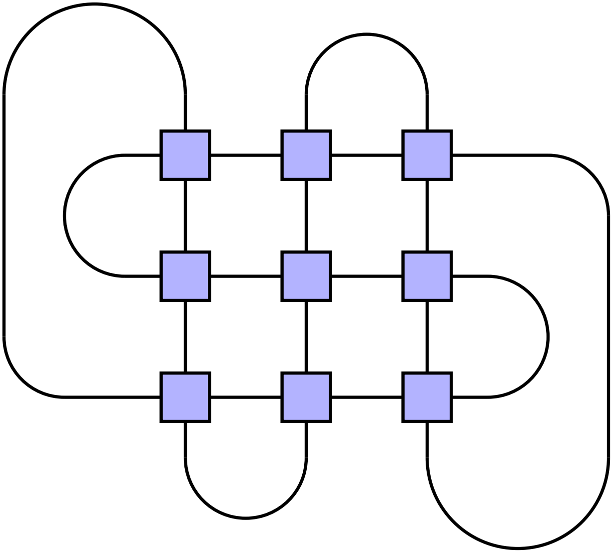
The Turaev surface - in pictures



An alternating example



An alternating example



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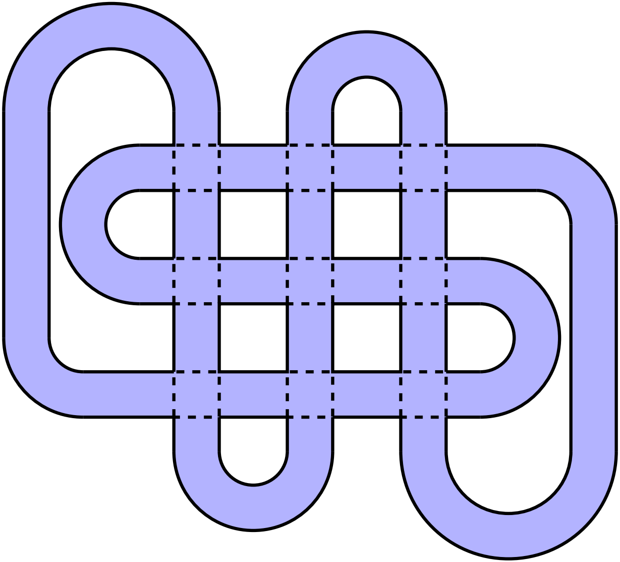

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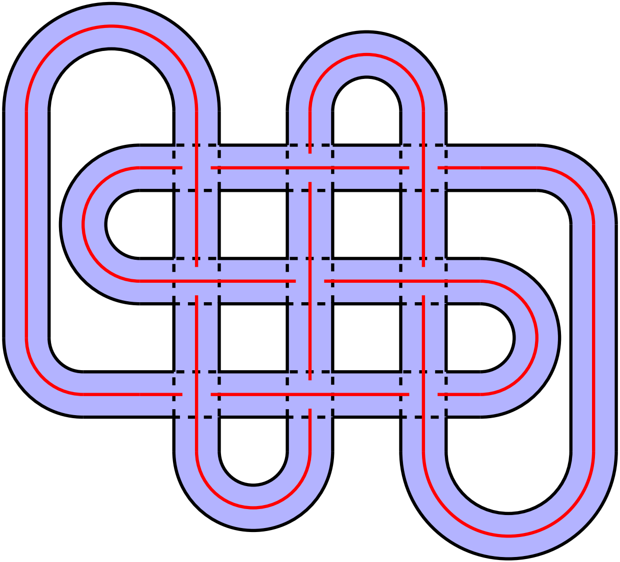
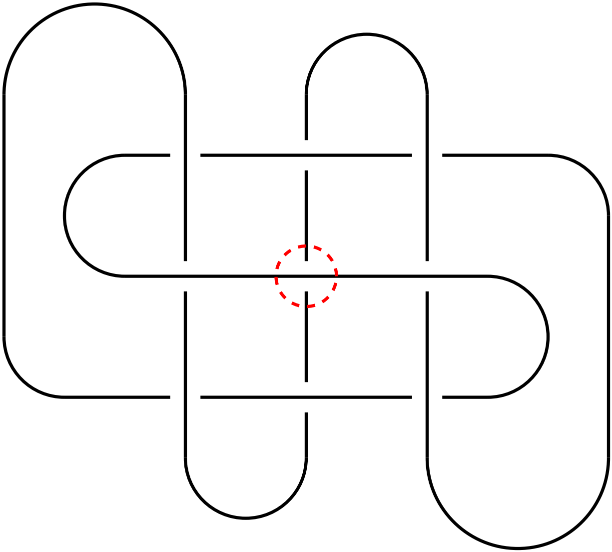


A non-alternating example



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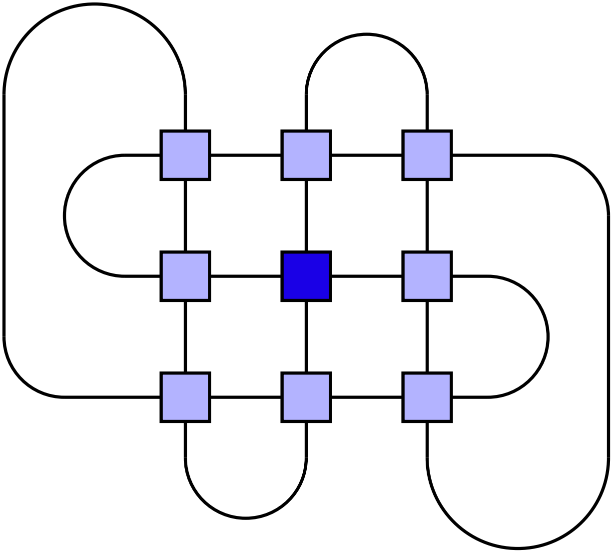

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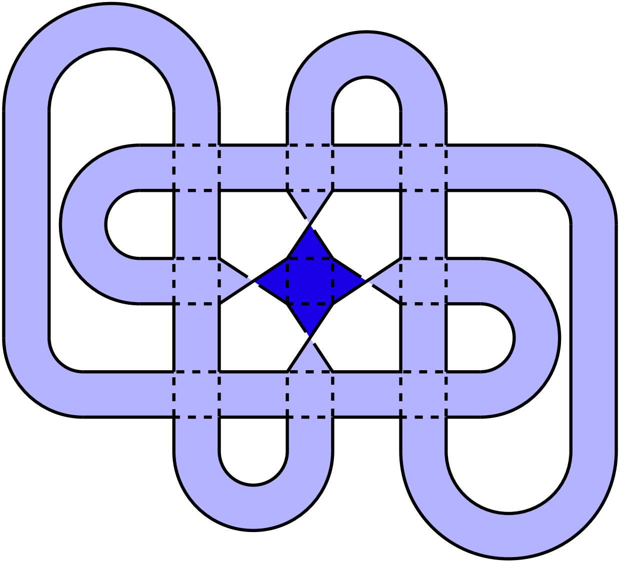

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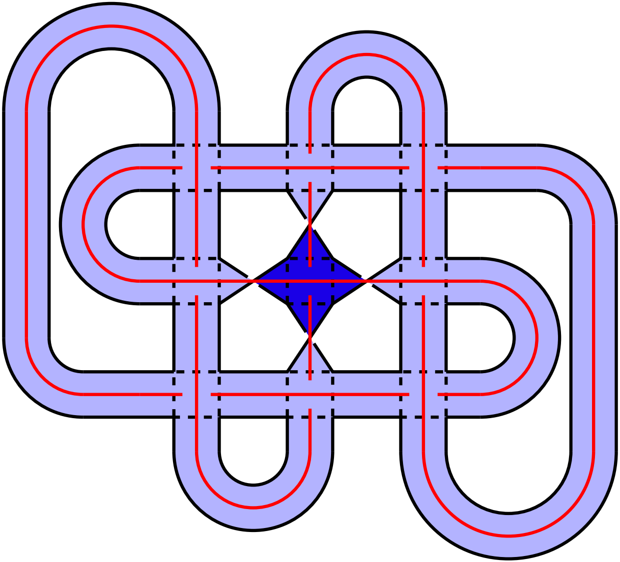


Turaev genus

- Define $g_T(D)$ to be the genus of the Turaev surface $F(D)$ of D.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

- $g_T(L) = 0$ if and only if L is alternating.

History

- To each link diagram D, Turaev (1987) associated a closed, oriented surface $F(D)$ of genus $g_T(D)$ and proved

$$c(L) - g_T(L) \geq \text{span } V_L(t).$$

- DFKLS (2006) - Further connections between $F(D)$ and the Jones polynomial $V_L(t)$.
- Champanerkar, Kofman, and Stoltzfus (2007) - Khovanov width gives lower bound for $g_T(L)$.
- L. (2008) - Knot Floer width gives lower bound for $g_T(L)$.
- Dasbach, L. (2011) - Further lower bounds for $g_T(L)$ coming from σ, τ, and s.
- Dasbach, L. (2014) - A Turaev surface model for Khovanov homology.

Properties of the Turaev surface

1. $F(D)$ is a Heegaard surface in S^3.
2. D is alternating on $F(D)$.
3. The complement of D in $F(D)$ is a collection of disks.

Question. If a surface satisfies (1) - (3) above, is it the Turaev surface of some knot?

Answer. No. There exist knots K of arbitrarily high Turaev genus that project to a Heegaard tori of S^3 in an alternating way and cut the torus into disks (L. - 2014).

Sufficient conditions for being a Turaev surface

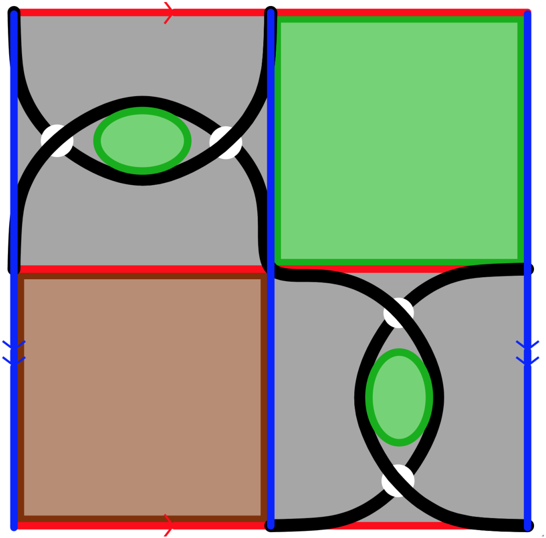
Suppose K is a knot and F is a surface satisfying

1. F is a Heegaard surface of S^3,
2. K has an alternating projection to F, and
3. the complement of the projection of K is a collection of disks.

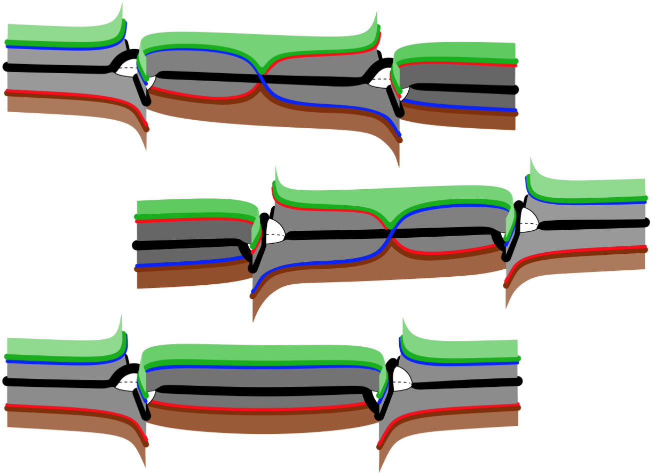
Two groups give additional conditions that ensure F is a Turaev surface:

- Champanerkar, Kofman (2014) - Conditions (1)-(3) plus a Morse theoretic condition.
- Armond, Druivenga, Kindred (2014) - Conditions (1)-(3) plus a Heegaard diagram condition.

Heegaard diagram corresponding to a Turaev surface



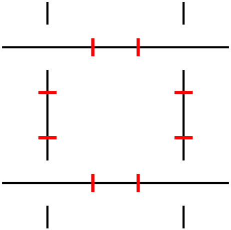
Heegaard curves on an edge of D

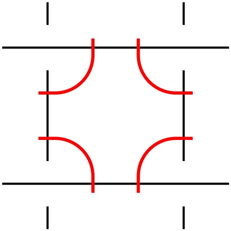


Alternating decompositions

- Consider D as a 4-regular plane graph with over/under information at vertices.
- Mark each non-alternating edge of D with two points.
- Connect marked points with non-intersecting arcs inside the faces of D that follow along the boundary of each face.
- The resulting collection of curves is the *alternating decomposition* of D (Thistlethwaite - 1988).

Arcs following the boundary of a face





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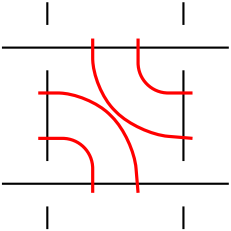
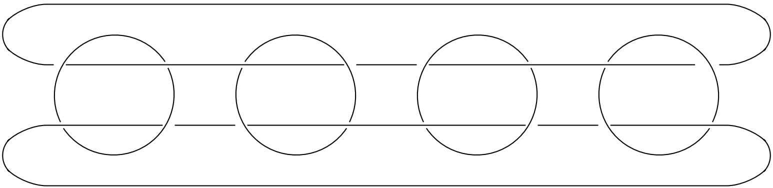


Bad

Example 1



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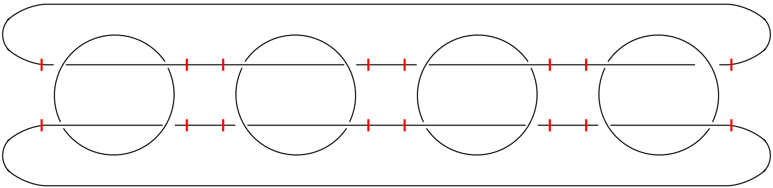

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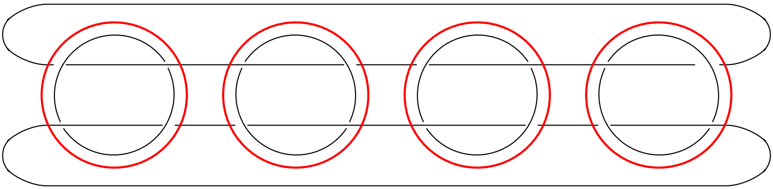
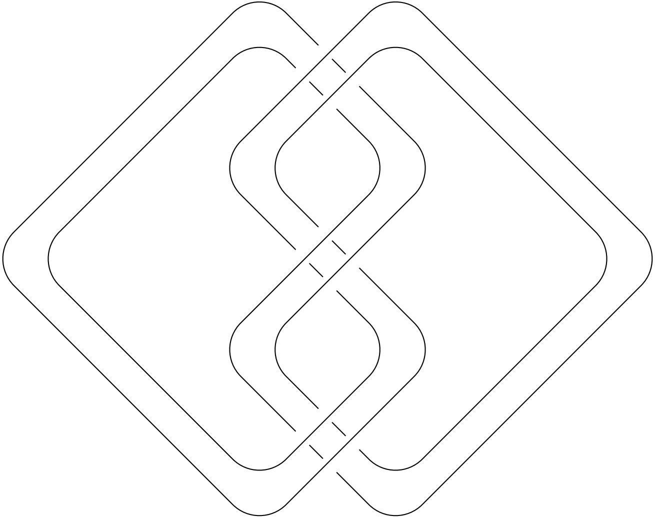
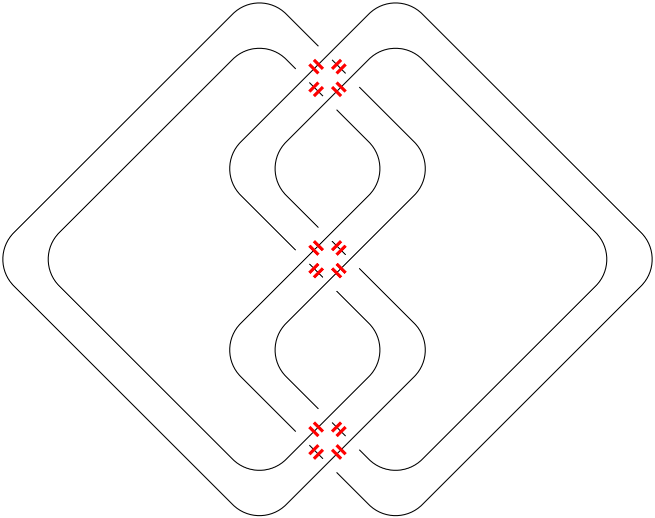


Example 2



Example 2



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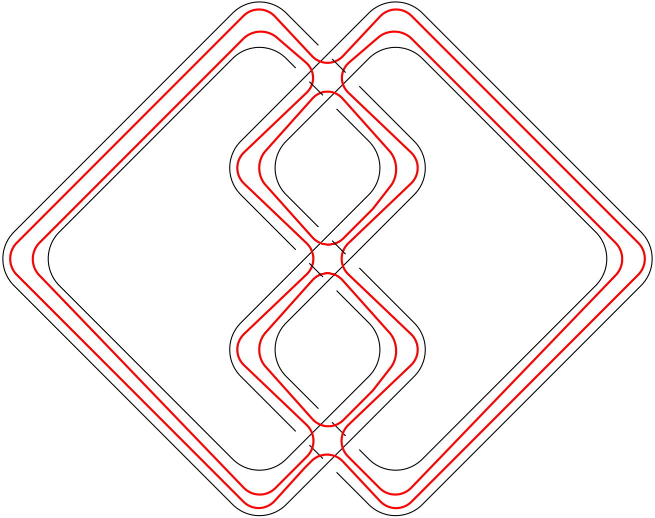
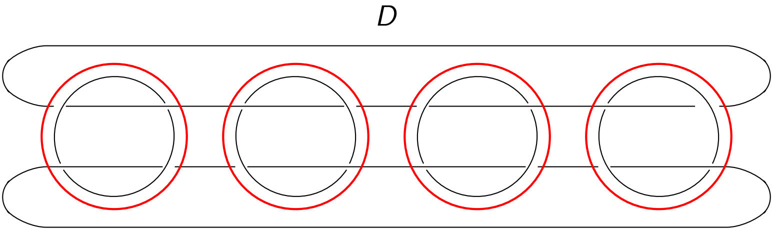
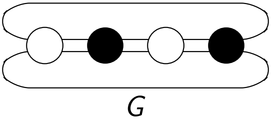


Alternating decomposition graph

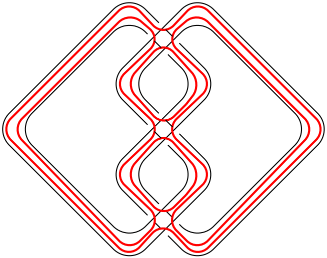
- The alternating decomposition curves partition S^2 into regions in which D is either alternating or has no crossings.
- Exactly one of the two regions adjacent to any alternating decomposition curve contains crossings.
- Form a plane graph G by taking the curves of the alternating decomposition to be the vertices of G and the edges of G to be the non-alternating edges of D.
- Each split alternating component of D are assigned is an isolated vertex.
- G is called the alternating decomposition graph of D.

Example 1

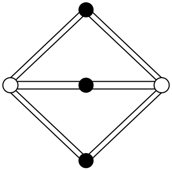




Example 2



D

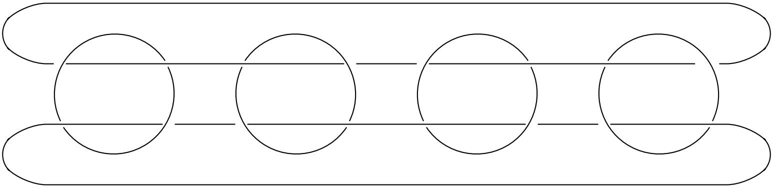


G

Properties of alternating decomposition graphs

- Alternating decomposition graphs are plane, bipartite, and each vertex has even degree.
- If D and D' have the same alternating decomposition graph, then $g_T(D) = g_T(D')$.
- If G is a plane, bipartite graph where every vertex has even degree, then G is an alternating decomposition graph of some link diagram D.
- Define $g_T(G) = g_T(D)$.

Example 1



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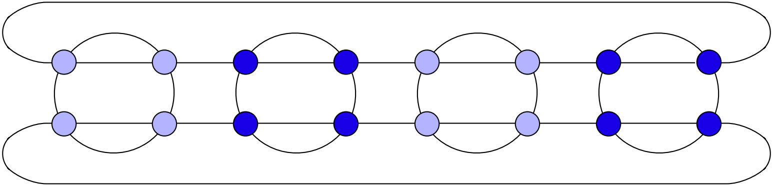

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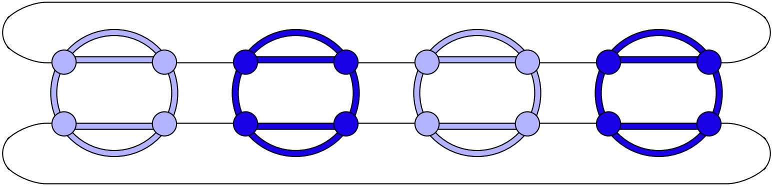

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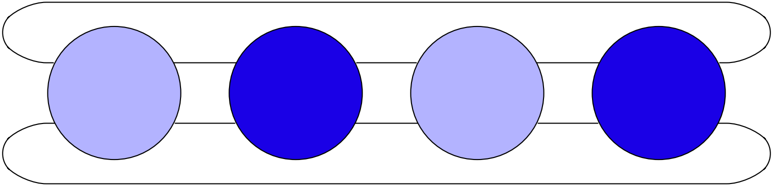

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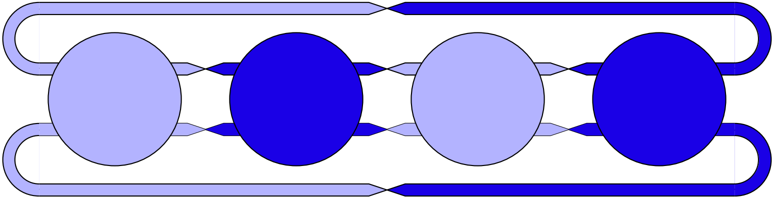


Graphs instead of diagrams

- There is a recursive algorithm for computing $g_T(G)$ without ever referring to a link diagram.
- $g_T(G)$ does not depend on the embedding of G into the plane.
- The recursive algorithm and a classification of Turaev genus zero diagrams/alternating decomposition graphs lead to the following theorems.

Turaev genus one diagrams

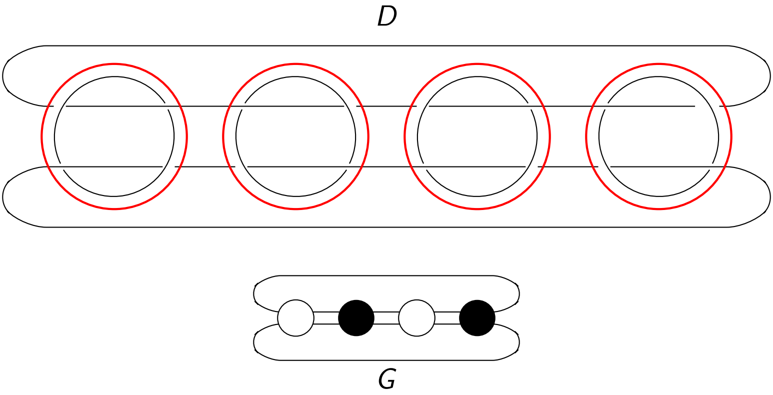
Theorem (Armond - L., Seungwon Kim)

Let L be a prime, non-split link with $g_T(L) = 1$. Every diagram D of L with $g_T(D) = 1$ has an alternating decomposition graph G that is a doubled cycle of even length.

Rephrasing. Every prime link diagram D with $g_T(D) = 1$ is a cycle of alternating 2-tangles.

Note. There is a version of this theorem for composite and split links.

Recall Example 1

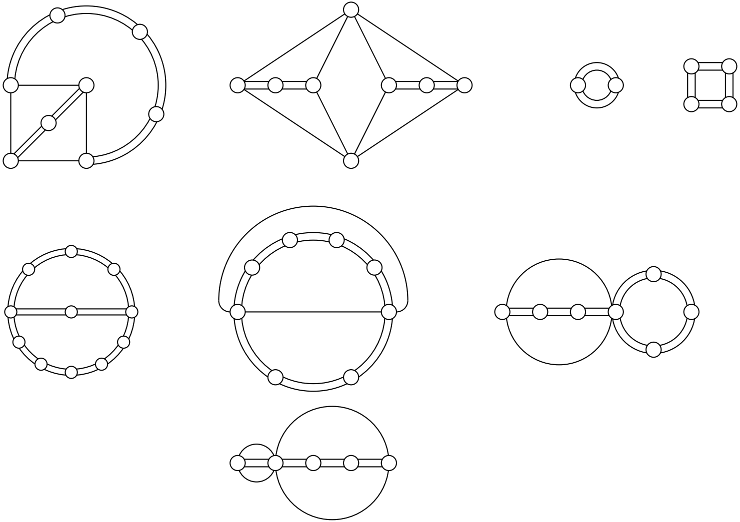


Turaev genus two diagrams

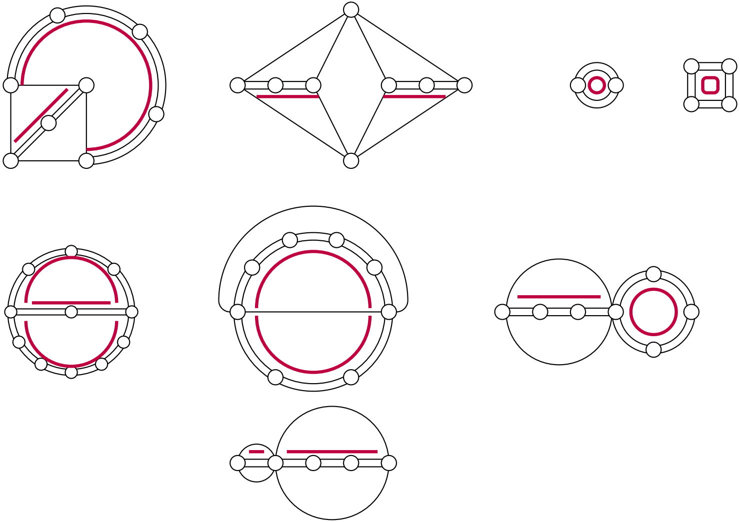
Theorem (Armond - L.)

Let D be a Turaev genus two diagram of a prime link with alternating decomposition graph G. Then G is among one of the seven infinite families appearing on the following slide.

Turaev genus two alternating decomposition graphs



Turaev genus two alternating decomposition graphs



Thank you!