Alternating tangle decompositions

Adam Lowrance - Vassar College

September 18, 2015

A common topological strategy

Suppose that we have a topological object X and want to compute a topological invariant $\text{Inv}(X)$.

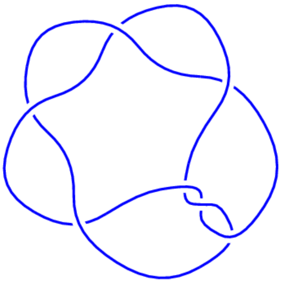
- Cut X into “nice” pieces $X_1, X_2, \ldots, X_k$.
- Compute $\text{Inv}(X_1), \text{Inv}(X_2), \ldots, \text{Inv}(X_k)$.
- Combine $\text{Inv}(X_1), \text{Inv}(X_2), \ldots, \text{Inv}(X_k)$ to recover $\text{Inv}(X)$.

How we will use this strategy

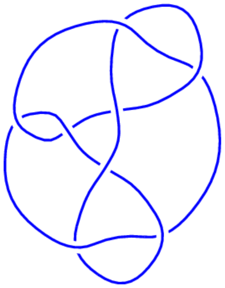
- Our topological object X will be a link or link diagram.
- Our topological invariants $\text{Inv}(X)$ will be Turaev genus and signature.
- Our "nice" pieces will be maximally alternating regions of the link diagram.

Alternating links

A link diagram is *alternating* if the crossings alternate over, under, over, under, ... as one travels along each component of the link. A link is *alternating* if it has an alternating diagram.



Alternating



Non-alternating

Why are alternating links nice?

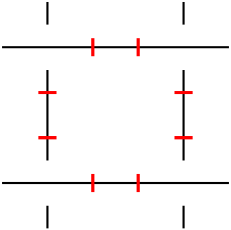
Alternating links have

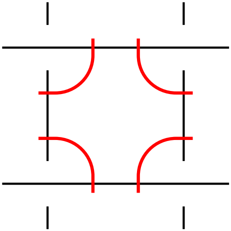
- computable crossing numbers,
- easily computable signatures,
- easily computable link polynomials (and generalizations),
- complements with well-understood hyperbolic structures, and
- many, many more nice properties.

Alternating tangle decompositions

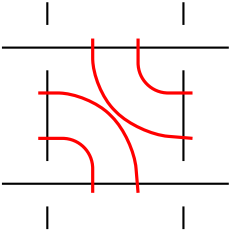
- Consider D as a 4-regular plane graph with over/under information at vertices.
- Mark each non-alternating edge of D with two points.
- Connect marked points with non-intersecting arcs inside the faces of D that follow along the boundary of each face.
- The resulting collection of curves is the *alternating tangle decomposition* of D (Thistlethwaite - 1988).

Arcs following the boundary of a face



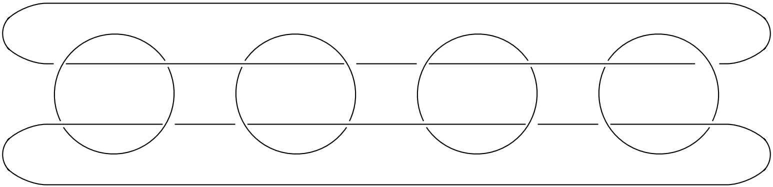


Good

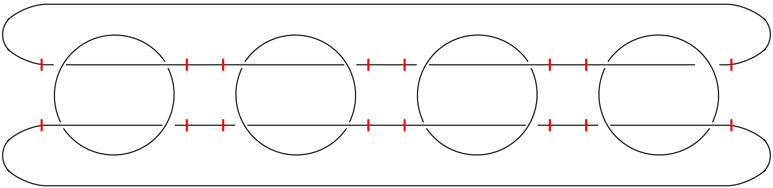


Bad

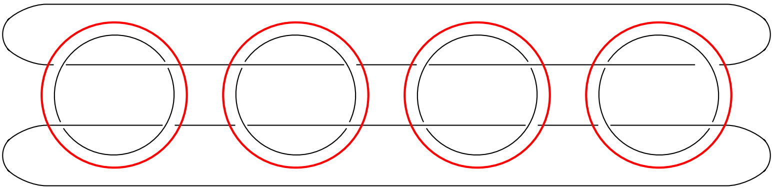
Example 1



Example 1



Example 1

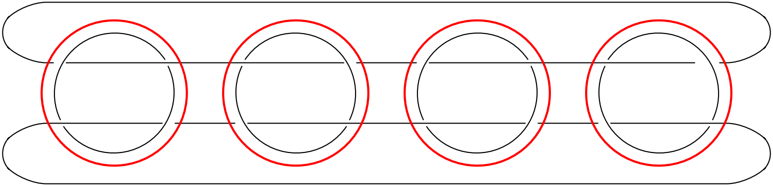


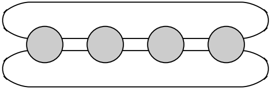
Alternating decomposition graph

- The alternating tangle decomposition curves partition S^2 into regions in which D is either alternating or has no crossings.
- Exactly one of the two regions adjacent to any alternating decomposition curve contains crossings.
- Form a plane graph G by taking the curves of the alternating tangle decomposition to be the vertices of G and the edges of G to be the non-alternating edges of D.
- Each split alternating component of D are assigned is an isolated vertex.
- G is called the alternating decomposition graph of D.

Example 1

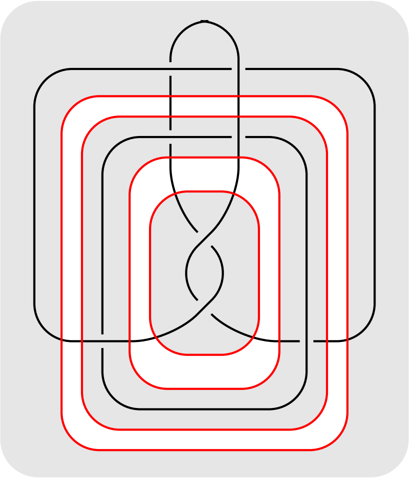
D



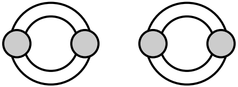


G

Example 2



D

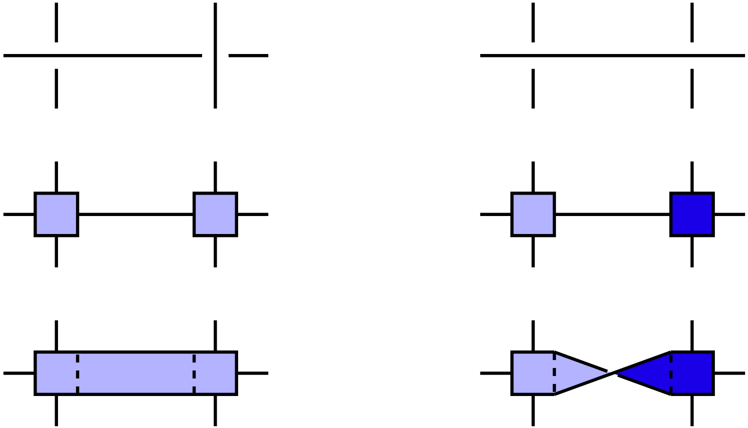


G

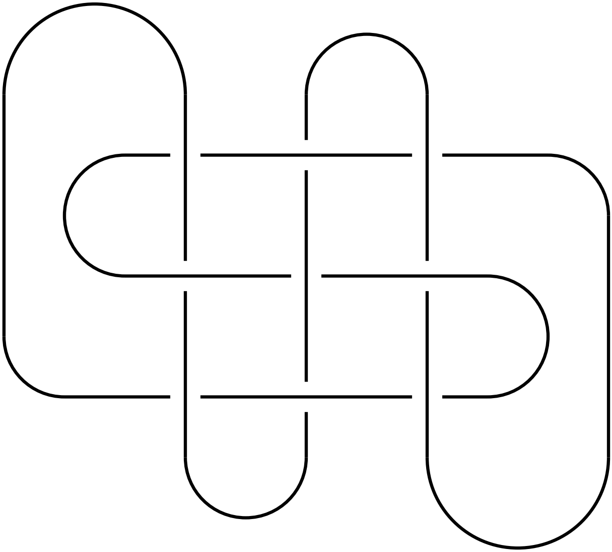
Construction of the Turaev surface $F(D)$

1. Replace crossings of D with disks.
2. Replace strands of D between crossings with (sometimes twisted) bands.
3. Cap off the boundary components with disks to obtain $F(D)$.

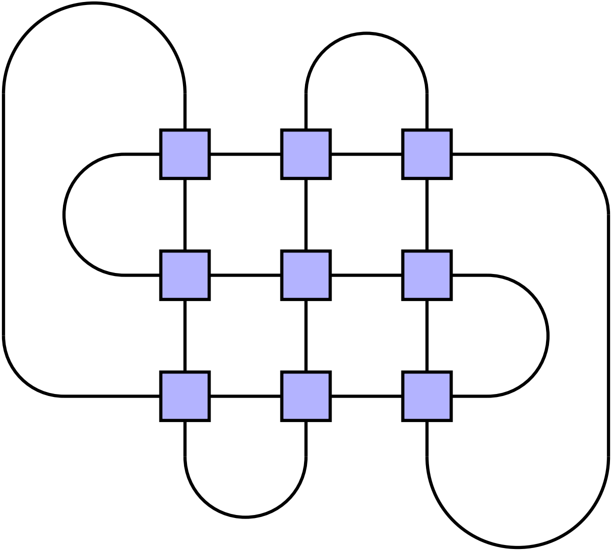
The Turaev surface - in pictures



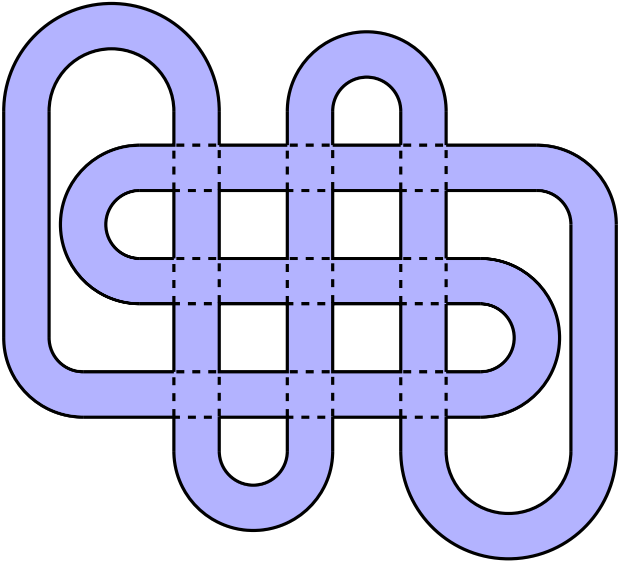
An alternating example



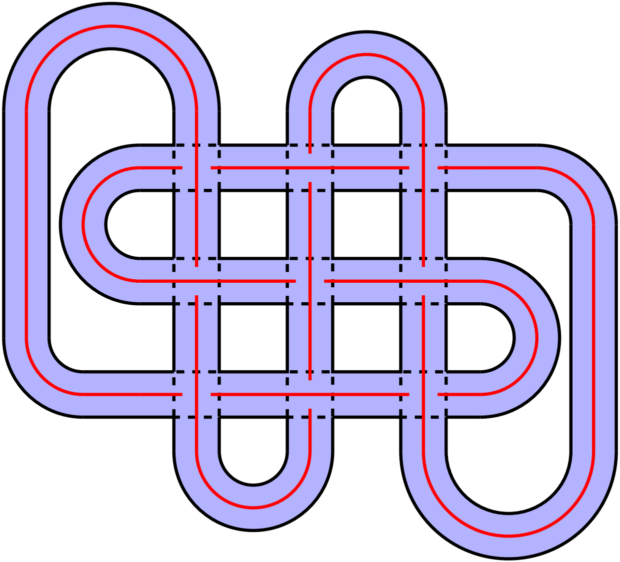
An alternating example



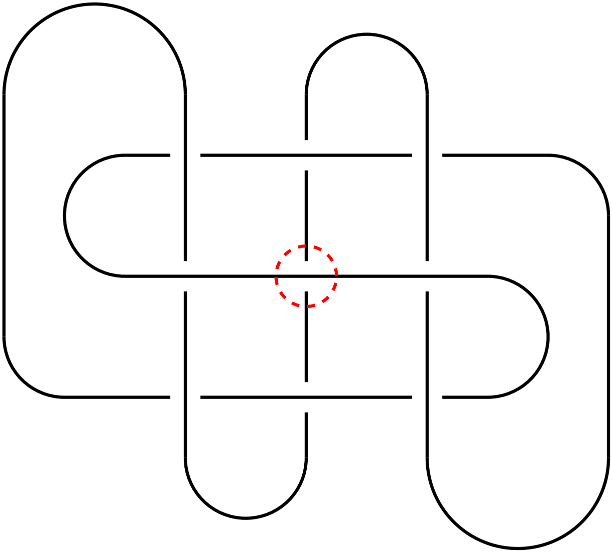
An alternating example



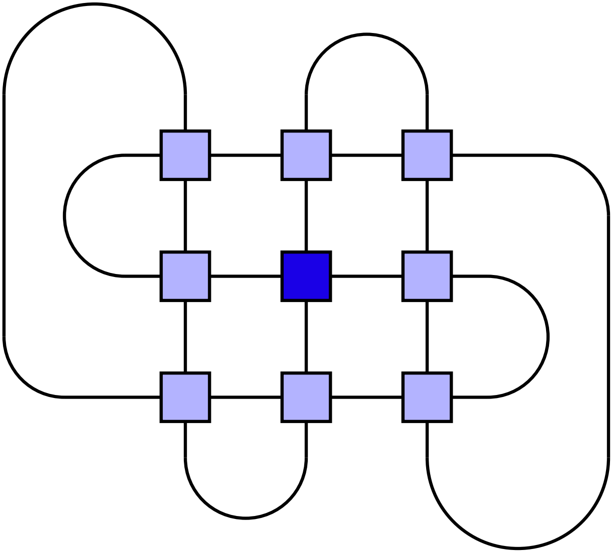
An alternating example



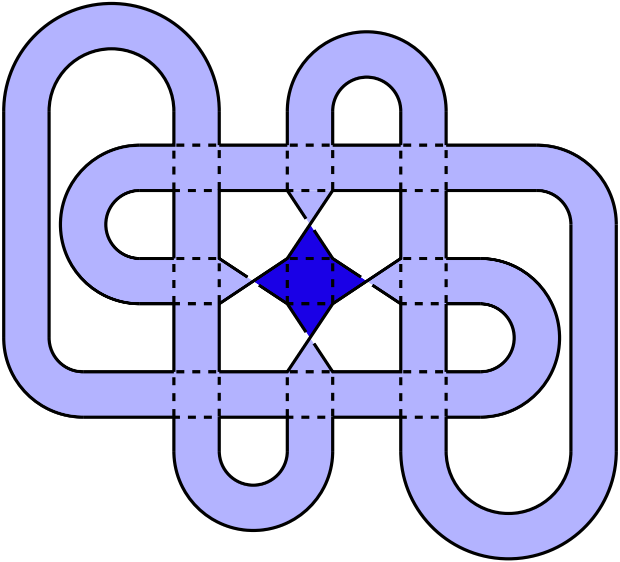
A non-alternating example



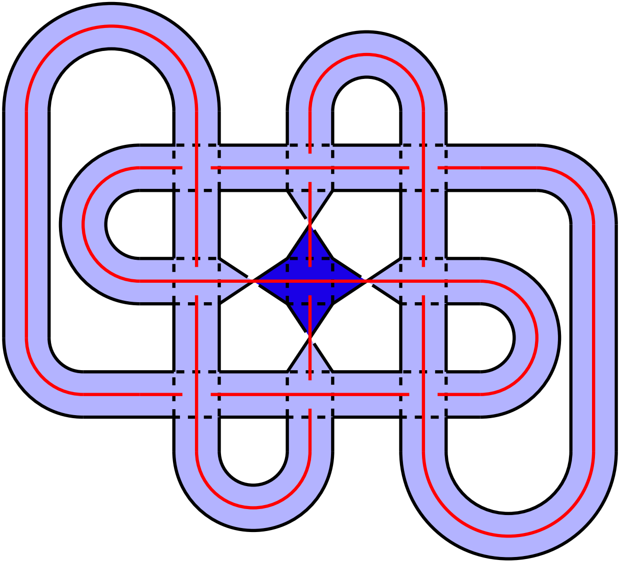
A non-alternating example



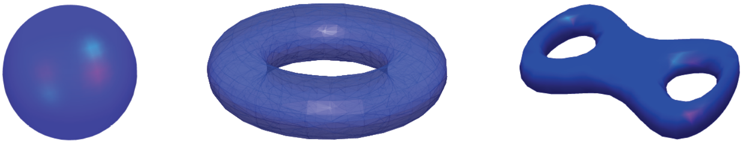
A non-alternating example



A non-alternating example



The genus of a surface



(a) sphere (b) torus (c) 2-torus

Turaev genus

- Define $g_T(D)$ to be the genus of the Turaev surface $F(D)$.
- The Turaev genus $g_T(L)$ of the link L is

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

- $g_T(L) = 0$ if and only if L is alternating.

History

- Turaev (1987) shows the genus of the Turaev surface has a relationship to the span of the Jones polynomial:

$$c(L) - g_T(L) \geq \text{span } V_L(t).$$

- DFKLS (2006) - Further connections between $F(D)$ and the Jones polynomial $V_L(t)$.
- Champanerkar, Kofman, and Stoltzfus (2007) - Khovanov width gives lower bound for $g_T(L)$.
- L. (2008) - Knot Floer width gives lower bound for $g_T(L)$.
- Dasbach, L. (2011) - Further lower bounds for $g_T(L)$ coming from σ, τ, and s.
- Dasbach, L. (2014) - A Turaev surface model for Khovanov homology.

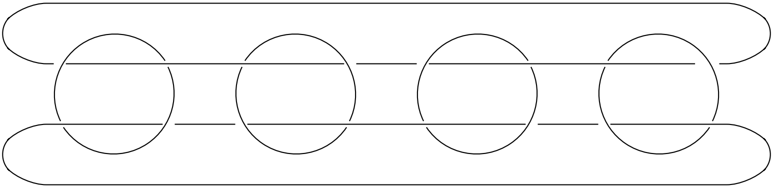
Alternating decomposition graph and Turaev genus

Theorem (Armond, L. - 2015)

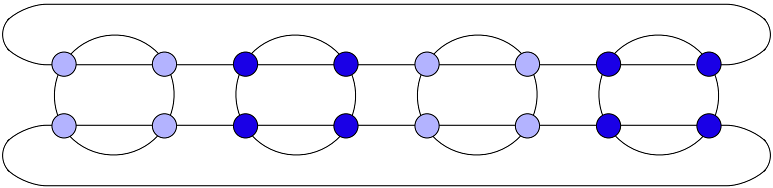
The alternating decomposition graph of D determines $g_T(D)$.

Proof idea. The pieces of the Turaev surface inside of the alternating regions are disks. The Turaev surface is obtained by gluing these disks together with twisted bands, then capping off boundary components.

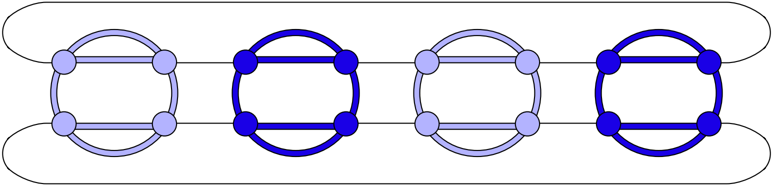
Proof by picture



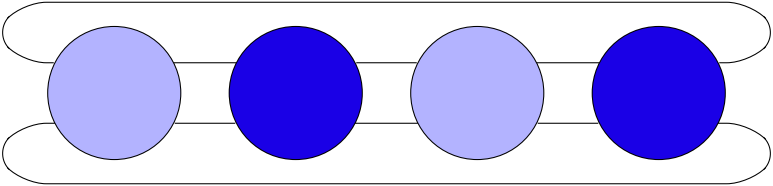
Proof by picture



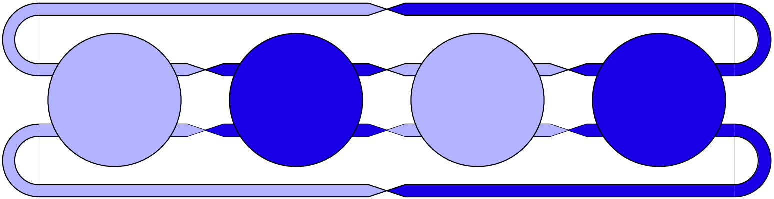
Proof by picture



Proof by picture



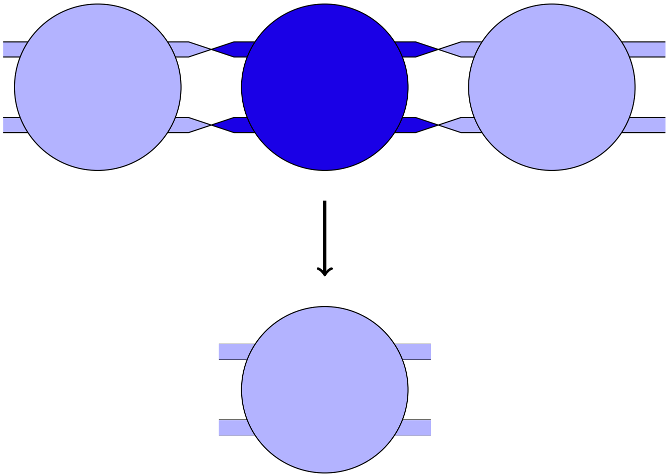
Proof by picture



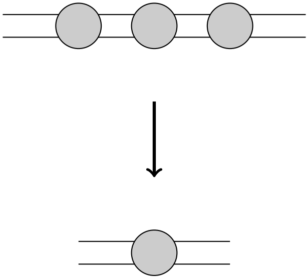
Classification goal

Goal: Classify link diagrams whose Turaev surface is a fixed number using alternating decomposition graphs.

A genus preserving move



A genus preserving move



A general classification theorem

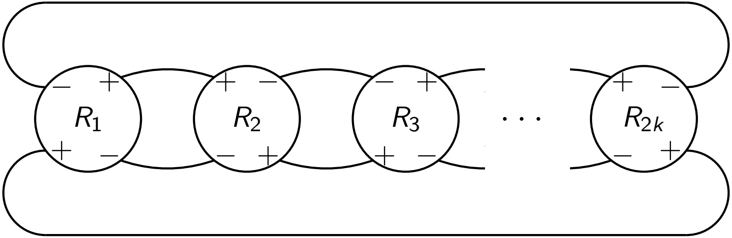
Theorem (Armond, L. - 2015)

For each non-negative integer n, there exists a finite set of graphs $G_1, \dots, G_k$ satisfying the following property. Every non-split link L with $g_T(L) \leq n$ has a diagram D with alternating decomposition graph G such that G can be transformed into G_i by genus preserving moves for some $i = 1, \dots, k$.

The Turaev genus one case

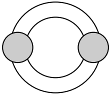
Theorem (Armond, L. - 2015, Kim - 2015)

Every link L with $g_T(L) \leq 1$ has a diagram as below, where each R_i is an alternating tangle.



The Turaev genus one case

In other words, every non-split, Turaev genus one link has a diagram whose alternating decomposition graph can be reduced to G via genus preserving moves.

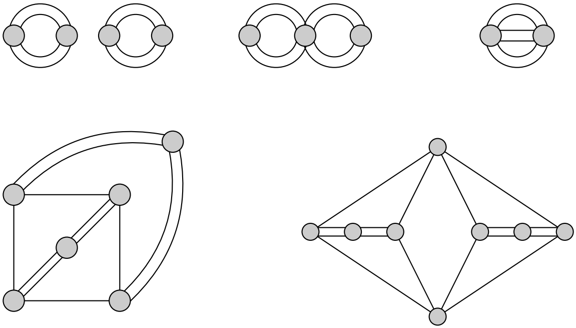


G

The Turaev genus two case

Theorem (Armond, L. - 2015 and Kim - 2015)

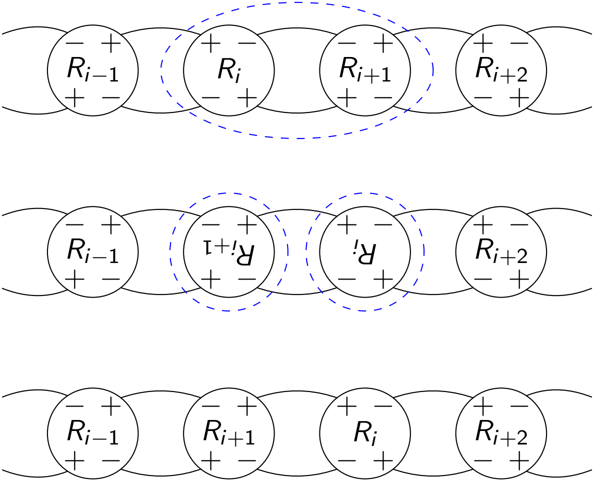
Every link with $g_T(L) \leq 2$ has a diagram whose alternating decomposition graph G can be transformed into one of the following graphs through genus preserving moves.



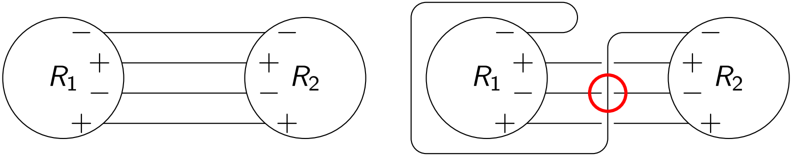
Almost alternating links

- An *almost alternating link* is a non-alternating link with a diagram where one crossing change transforms it into an alternating diagram (Adams, Brock, Bugbee, Comar, Faigin, Huston, Joseph, Pesikoff - 1992).
- Almost alternating links have Turaev genus one (Abe - 2008).
- Non-adequate links of Turaev genus one are almost alternating (Kim - 2015).
- Every Turaev genus one link is mutant to an almost alternating link (Armond, L. - 2015).

Mutation proof



Mutation proof continued

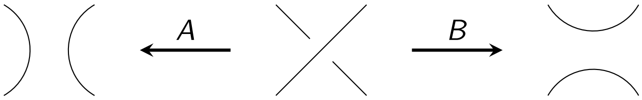


Link signature

- The signature $\sigma(L)$ of a link was defined in various forms by Trotter (1962), Murasugi (1965), Milnor (1968), and Erle (1969).
- The signature of a link is the signature of the Seifert matrix M of the link, that is the difference between the number of positive and negative eigenvalues of M.
- The signature of a knot $\sigma(K)$ gives a lower bound on the unknotting number $u(K)$ of K:

$$|\sigma(K)| \leq 2u(K).$$

A and B resolutions



The A and B resolutions of a crossing.

- A choice of resolutions at each crossing is called a *Kauffman state*.
- Let $s_A(D)$ and $s_B(D)$ be the number of components in the all-A state and all-B state respectively.

Bounds on signature of links

$+$ $\qquad$ $-$

- Let $c_+(D)$ and $c_-(D)$ be the number of positive and negative crossings in D.
- (Traczyk - 2004) If D is an alternating diagram of the link L, then

$$\sigma(L) = s_A(D) - c_+(D) - 1 = -s_B(D) + c_-(D) + 1.$$

- (Dasbach, L. - 2010) For any link L with diagram D, the following inequality holds:

$$s_A(D) - c_+(D) - 1 \leq \sigma(L) \leq -s_B(D) + c_-(D) + 1.$$

The signature of Turaev genus one knots

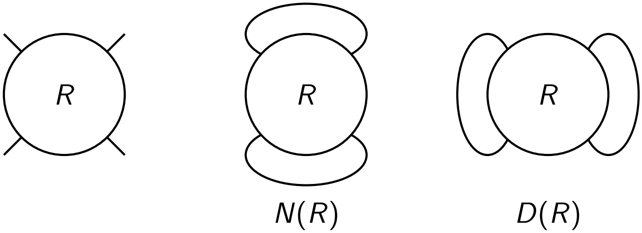
The *determinant* $\det(K)$ of a knot K is $\det(K) = |\Delta_K(-1)|$ where $\Delta_K(t)$ is the Alexander polynomial of K.

Theorem (Dasbach, L. - 2015)

Let K be a knot with diagram D whose Turaev surface has genus one. The signature of K is determined by

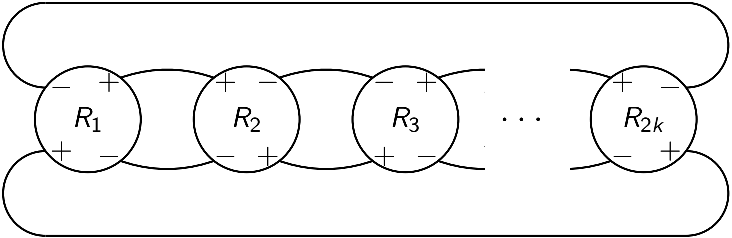
$$\sigma(K) = s_A(D) - c_+(D) \pm 1 \text{ and } \sigma(K) \equiv \det(K) - 1 \mod 4.$$

Tangle closures



The tangle R, its numerator closure $N(R)$, and its denominator closure $D(R)$.

Signatures and alternating tangle decompositions



Theorem (Dasbach, L. - 2015)

Let L be a link with Turaev genus one diagram as above. Then

$$\sigma(L) = \begin{cases} \pm 1 + \sum_{i=1}^{2k} \sigma(N(R_i)) \\ \pm 1 + \sum_{i=1}^{2k} \sigma(D(R_i)), \end{cases}$$

where the choice of numerator or denominator is determined by the orientation of the link.

Open Questions

- Does our formula for the signature of Turaev genus one knots allow us to compute unknotting numbers of any knots?
- Can we (approximately) express the signature of a link in terms of the signatures of the closures of the tangles in its alternating tangle decomposition?
- Is every link of Turaev genus one almost alternating?

Thank you!